

Holonomic Techniques, Periods, and Decision Problems

Joël Ouaknine

Max Planck Institute for Software Systems

(joint work w/ George Kenison, Oleksiy Klurman, Engel Lefauchaux, Florian Luca, Pieter Moree, Eike Neumann, Emre Sertöz, Markus Whiteland, James Worrell)

Kolchin Seminar in Differential Algebra
5 December 2025





Science is what we understand well
enough to explain to a computer.
Art is everything else we do.

— *Donald Knuth* —

AZ QUOTES

What do all these things have in common ...?

- High-performance algorithms for computing π , $\frac{1}{\pi}$, e , γ , ...
- The irrationality of $\zeta(3)$
- Bessel function identities
- Calabi-Yau parameterizations
- Enumeration and asymptotics in statistical mechanics
- Automated symbolic integration
- The cost of searching in quadtrees, m-ary search trees
- Binomial identities
- Random walks in the plane
- Young tableaux of bounded height
- Hypergeometric functions
- Latin squares
- The Jacobi triple product identity
- Alternating sign matrices
- Consecutive records in permutations
- Noncrossing partitions

(shamelessly stolen from Cyril Banderier)

Answer: HOLONOMIC TECHNIQUES!

- High-performance algorithms for computing π , $\frac{1}{\pi}$, e , γ , $\dots$
- The irrationality of $\zeta(3)$
- Bessel function identities
- Calabi-Yau parameterizations
- Enumeration and asymptotics in statistical mechanics
- Automated symbolic integration
- The cost of searching in quadtrees, m -ary search trees
- Binomial identities
- Random walks in the plane
- Young tableaux of bounded height
- Hypergeometric functions
- Latin squares
- The Jacobi triple product identity
- Alternating sign matrices
- Consecutive records in permutations
- Noncrossing partitions

(shamelessly stolen from Cyril Banderier)

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

of ways to choose n letters
among $2n$ letters:

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

of ways to choose n letters
among $2n$ letters:

$$\underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

choose k among n
 $\times$
choose $(n - k)$ among n

$$\underbrace{A_1 A_2 \dots A_n}_{\binom{n}{k}} \times \underbrace{B_1 B_2 \dots B_n}_{\binom{n}{n-k}}$$

of ways to choose n letters
among $2n$ letters:

$$\underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Sum over all k :

choose k among n

$\times$

choose $(n - k)$ among n

$$\sum_{k=0}^n \underbrace{A_1 A_2 \dots A_n}_{\binom{n}{k}} \times \underbrace{B_1 B_2 \dots B_n}_{\binom{n}{n-k}}$$

of ways to choose n letters
among $2n$ letters:

$$\underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Sum over all k :

choose k among n

$\times$

choose $(n - k)$ among n

$$\sum_{k=0}^n \underbrace{A_1 A_2 \dots A_n}_{\binom{n}{k}} \times \underbrace{B_1 B_2 \dots B_n}_{\binom{n}{n-k}}$$

of ways to choose n letters
among $2n$ letters:

$$\underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

$=$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Sum over all k :

choose k among n

$\times$

choose $(n - k)$ among n

of ways to choose n letters
among $2n$ letters:

$$\sum_{k=0}^n \underbrace{A_1 A_2 \dots A_n}_{\binom{n}{k}} \times \underbrace{B_1 B_2 \dots B_n}_{\binom{n}{n-k}} = \underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

Thus $\sum_{k=0}^n \binom{n}{k} \binom{n}{n-k}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Sum over all k :

choose k among n

$\times$

choose $(n - k)$ among n

of ways to choose n letters
among $2n$ letters:

$$\sum_{k=0}^n \underbrace{A_1 A_2 \dots A_n}_{\binom{n}{k}} \times \underbrace{B_1 B_2 \dots B_n}_{\binom{n}{n-k}} = \underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

Thus $\sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^n \binom{n}{k}^2$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Sum over all k :

choose k among n

$\times$

choose $(n - k)$ among n

of ways to choose n letters
among $2n$ letters:

$$\sum_{k=0}^n \underbrace{A_1 A_2 \dots A_n}_{\binom{n}{k}} \times \underbrace{B_1 B_2 \dots B_n}_{\binom{n}{n-k}} = \underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

Thus $\sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad \square$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Sum over all k :

choose k among n

×

choose $(n - k)$ among n

of ways to choose n letters
among $2n$ letters:

$$\sum_{k=0}^n \underbrace{A_1 A_2 \dots A_n}_{\binom{n}{k}} \times \underbrace{B_1 B_2 \dots B_n}_{\binom{n}{n-k}} = \underbrace{A_1 A_2 \dots A_n B_1 B_2 \dots B_n}_{\binom{2n}{n}}$$

Thus $\sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n} \quad \square$



Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

High-level strategy:

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

High-level strategy:

- 1 Find a recurrence relation for the sequence $\langle u_n \rangle_{n=0}^{\infty}$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

High-level strategy:

- 1 Find a recurrence relation for the sequence $\langle u_n \rangle_{n=0}^{\infty}$
- 2 Find a recurrence relation for the sequence $\langle v_n \rangle_{n=0}^{\infty}$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

High-level strategy:

- 1 Find a recurrence relation for the sequence $\langle u_n \rangle_{n=0}^{\infty}$
- 2 Find a recurrence relation for the sequence $\langle v_n \rangle_{n=0}^{\infty}$
- 3 Combine to obtain a recurrence relation for $\langle u_n - v_n \rangle_{n=0}^{\infty}$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

High-level strategy:

- 1 Find a recurrence relation for the sequence $\langle u_n \rangle_{n=0}^{\infty}$
- 2 Find a recurrence relation for the sequence $\langle v_n \rangle_{n=0}^{\infty}$
- 3 Combine to obtain a recurrence relation for $\langle u_n - v_n \rangle_{n=0}^{\infty}$
- 4 Finally, verify that $u_n - v_n \equiv 0$ by checking the initial values

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Write $v_n = \binom{2n}{n} = \frac{(2n)!}{n!n!}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Write $v_n = \binom{2n}{n} = \frac{(2n)!}{n!n!}$

So $v_{n+1} = \binom{2n+2}{n+1}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Write $v_n = \binom{2n}{n} = \frac{(2n)!}{n!n!}$

So $v_{n+1} = \binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Write $v_n = \binom{2n}{n} = \frac{(2n)!}{n!n!}$

So $v_{n+1} = \binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)n!(n+1)n!}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Write $v_n = \binom{2n}{n} = \frac{(2n)!}{n!n!}$

So $v_{n+1} = \binom{2n+2}{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)n!(n+1)n!}$

Thus:

$$\begin{aligned}(n+1)^2 v_{n+1} &= (2n+2)(2n+1)v_n \\ v_0 &= 1\end{aligned}$$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Let $u_n = \sum_{k=0}^n \binom{n}{k}^2$

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Let $u_n = \sum_{k=0}^n \binom{n}{k}^2$

? ? ?

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Let $u_n = \sum_{k=0}^n \binom{n}{k}^2$

? ? ?

$\binom{n}{k}^2$ is **Sister-Celine summable!** *[details hidden ...]*

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Let $u_n = \sum_{k=0}^n \binom{n}{k}^2$

? ? ?

$\binom{n}{k}^2$ is **Sister-Celine summable!** *[details hidden ...]*

Use **Zeilberger's Algorithm** to get:

Holonomic Techniques: A Simple Example

Prove that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Let $u_n = \sum_{k=0}^n \binom{n}{k}^2$

? ? ?

$\binom{n}{k}^2$ is **Sister-Celine summable!** *[details hidden ...]*

Use **Zeilberger's Algorithm** to get:

$$\begin{aligned}(n+1)u_{n+1} &= (4n+2)u_n \\ u_0 &= 1\end{aligned}$$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

$$(n+1)u_{n+1} = (4n+2)u_n$$

$$u_0 = 1$$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

$$\begin{aligned}(n+1)u_{n+1} &= (4n+2)u_n \\ u_0 &= 1\end{aligned}$$

$$\begin{aligned}(n+1)^2 v_{n+1} &= (2n+2)(2n+1)v_n \\ v_0 &= 1\end{aligned}$$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

$$\begin{aligned}(n+1)u_{n+1} &= (4n+2)u_n \\ u_0 &= 1\end{aligned}$$

$$\begin{aligned}(n+1)^2 v_{n+1} &= (2n+2)(2n+1)v_n \\ v_0 &= 1\end{aligned}$$

- $\langle u_n \rangle_{n=0}^\infty$ and $\langle v_n \rangle_{n=0}^\infty$ both satisfy a *holonomic recurrence relation* (linear recurrence relation with polynomial coefficients) of *order 1*

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

$$(n+1)u_{n+1} = (4n+2)u_n$$
$$u_0 = 1$$

$$(n+1)^2 v_{n+1} = (2n+2)(2n+1)v_n$$
$$v_0 = 1$$

- $\langle u_n \rangle_{n=0}^\infty$ and $\langle v_n \rangle_{n=0}^\infty$ both satisfy a *holonomic recurrence relation* (linear recurrence relation with polynomial coefficients) of *order 1*
- Therefore $\langle u_n - v_n \rangle_{n=0}^\infty$ satisfies a holonomic recurrence relation of order 2

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

$$\begin{aligned}(n+1)u_{n+1} &= (4n+2)u_n \\ u_0 &= 1\end{aligned}$$

$$\begin{aligned}(n+1)^2 v_{n+1} &= (2n+2)(2n+1)v_n \\ v_0 &= 1\end{aligned}$$

- $\langle u_n \rangle_{n=0}^\infty$ and $\langle v_n \rangle_{n=0}^\infty$ both satisfy a *holonomic recurrence relation* (linear recurrence relation with polynomial coefficients) of *order 1*
- Therefore $\langle u_n - v_n \rangle_{n=0}^\infty$ satisfies a holonomic recurrence relation of order 2
- Verify that $u_0 - v_0 = 0$ and $u_1 - v_1 = 0$

Holonomic Techniques: A Simple Example

Prove that $\underbrace{\sum_{k=0}^n \binom{n}{k}^2}_{u_n} = \underbrace{\binom{2n}{n}}_{v_n}$

$$\begin{aligned}(n+1)u_{n+1} &= (4n+2)u_n \\ u_0 &= 1\end{aligned}$$

$$\begin{aligned}(n+1)^2 v_{n+1} &= (2n+2)(2n+1)v_n \\ v_0 &= 1\end{aligned}$$

- $\langle u_n \rangle_{n=0}^\infty$ and $\langle v_n \rangle_{n=0}^\infty$ both satisfy a *holonomic recurrence relation* (linear recurrence relation with polynomial coefficients) of *order 1*
- Therefore $\langle u_n - v_n \rangle_{n=0}^\infty$ satisfies a holonomic recurrence relation of order 2
- Verify that $u_0 - v_0 = 0$ and $u_1 - v_1 = 0$
- Conclude that $u_n - v_n \equiv 0$



A More Complicated Example

In 1977, Roger Apéry produced a sensational proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$



A More Complicated Example

In 1977, Roger Apéry produced a sensational proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$



Crucial ingredient: the “Apéry numbers”

$A_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ obey the holonomic recurrence

$$(n+2)^3 A_{n+2} - (2n+3)(17n^2 + 51n + 39)A_{n+1} + (n+1)^3 A_n = 0$$

A More Complicated Example

In 1977, Roger Apéry produced a sensational proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$



Crucial ingredient: the “Apéry numbers”

$A_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ obey the holonomic recurrence

$$(n+2)^3 A_{n+2} - (2n+3)(17n^2 + 51n + 39)A_{n+1} + (n+1)^3 A_n = 0$$

Alf van der Poorten writes: *“Neither Cohen nor I had been able to prove [the above] in the intervening two months. . . .*



A More Complicated Example

In 1977, Roger Apéry produced a sensational proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$



Crucial ingredient: the “Apéry numbers”

$A_n := \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ obey the holonomic recurrence

$$(n+2)^3 A_{n+2} - (2n+3)(17n^2 + 51n + 39)A_{n+1} + (n+1)^3 A_n = 0$$

Alf van der Poorten writes: *“Neither Cohen nor I had been able to prove [the above] in the intervening two months. . . . After days of fruitless effort the specific problem was mentioned to Don Zagier, and with irritating speed he showed that indeed the sequence satisfies the recurrence. This more or less broke the dam and the rest of the proof was quickly conquered.”*



A More Complicated Example

Unfortunately van der Poorten
was a couple decades too early!!

A More Complicated Example

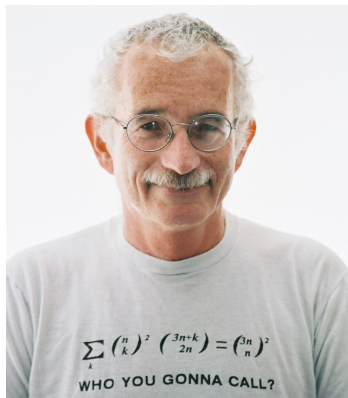
Unfortunately van der Poorten
was a couple decades too early!!

Nowadays one can use
Zeilberger's algorithm not only to
prove such a recurrence, but
even to **derive** it from scratch!

A More Complicated Example

Unfortunately van der Poorten was a couple decades too early!!

Nowadays one can use **Zeilberger's algorithm** not only to **prove** such a recurrence, but even to **derive** it from scratch!



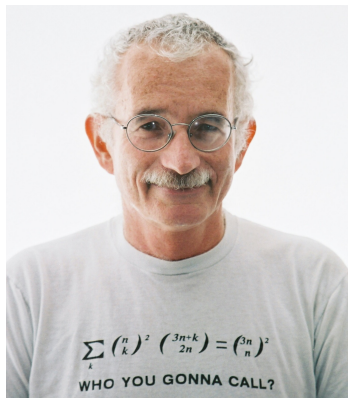
A More Complicated Example

Unfortunately van der Poorten
was a couple decades too early!!

Nowadays one can use
Zeilberger's algorithm not only to
prove such a recurrence, but
even to **derive** it from scratch!

The Maple SumTools command:

```
> sumrecursion(binomial(n,k)^2*binomial(n+k,k)^2,k,s(n));
```



A More Complicated Example

Unfortunately van der Poorten
was a couple decades too early!!

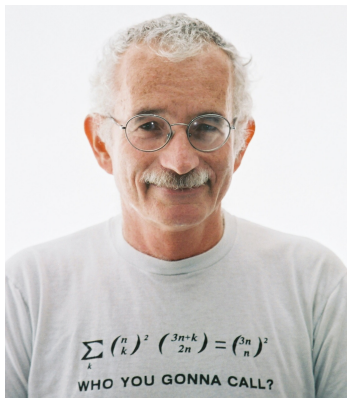
Nowadays one can use
Zeilberger's algorithm not only to
prove such a recurrence, but
even to **derive** it from scratch!

The Maple SumTools command:

```
> sumrecursion(binomial(n,k)^2*binomial(n+k,k)^2,k,s(n));
```

outputs **completely automatically**:

$$(n+2)^3 s(n+2) - (2n+3)(17n^2 + 51n + 39)s(n+1) + (n+1)^3 s(n) = 0$$



A Continuous Example

Prove that $\sin(2x) = 2 \sin(x) \cos(x)$

A Continuous Example

Prove that $\sin(2x) = 2 \sin(x) \cos(x)$

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$

A Continuous Example

Prove that $\underbrace{\sin(2x)}_{f(x)} = \underbrace{2 \sin(x) \cos(x)}_{g(x)}$

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$

A Continuous Example

Prove that $\underbrace{\sin(2x)}_{f(x)} = \underbrace{2 \sin(x) \cos(x)}_{g(x)}$

We have

$$4f(x) + f''(x) = 0$$

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$

A Continuous Example

Prove that $\underbrace{\sin(2x)}_{f(x)} = \underbrace{2 \sin(x) \cos(x)}_{g(x)}$

We have

$$4f(x) + f''(x) = 0$$

(i.e. $f(x)$ satisfies a second-order linear differential equation)

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$

A Continuous Example

Prove that $\underbrace{\sin(2x)}_{f(x)} = 2 \underbrace{\sin(x) \cos(x)}_{g(x)}$

We have

$$4f(x) + f''(x) = 0$$

(i.e. $f(x)$ satisfies a second-order linear differential equation)

$g(x)$ is the product of two functions, each of which satisfies a second-order linear ODE; so $g(x)$ has order at most 4

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$

A Continuous Example

Prove that $\underbrace{\sin(2x)}_{f(x)} = \underbrace{2 \sin(x) \cos(x)}_{g(x)}$

We have

$$4f(x) + f''(x) = 0$$

(i.e. $f(x)$ satisfies a second-order linear differential equation)

$g(x)$ is the product of two functions, each of which satisfies a second-order linear ODE; so $g(x)$ has order at most 4

Thus $h(x) \equiv f(x) - g(x)$ satisfies a linear ODE of order at most 6

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$

A Continuous Example

Prove that $\underbrace{\sin(2x)}_{f(x)} = \underbrace{2 \sin(x) \cos(x)}_{g(x)}$

We have

$$4f(x) + f''(x) = 0$$

(i.e. $f(x)$ satisfies a second-order linear differential equation)

$g(x)$ is the product of two functions, each of which satisfies a second-order linear ODE; so $g(x)$ has order at most 4

Thus $h(x) \equiv f(x) - g(x)$ satisfies a linear ODE of order at most 6

Verify that $h(0) = h'(0) = h''(0) = \dots = h^{(5)}(0) = 0$

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$

A Continuous Example

Prove that $\underbrace{\sin(2x)}_{f(x)} = \underbrace{2 \sin(x) \cos(x)}_{g(x)}$

We have

$$4f(x) + f''(x) = 0$$

(i.e. $f(x)$ satisfies a second-order linear differential equation)

$g(x)$ is the product of two functions, each of which satisfies a second-order linear ODE; so $g(x)$ has order at most 4

Thus $h(x) \equiv f(x) - g(x)$ satisfies a linear ODE of order at most 6

Verify that $h(0) = h'(0) = h''(0) = \dots = h^{(5)}(0) = 0$

Conclude that $h(x) \equiv 0$

What we're assuming:

$$\sin''(x) = -\sin(x)$$

$$\sin(0) = 0 ; \sin'(0) = 1$$

$$\cos''(x) = -\cos(x)$$

$$\cos(0) = 1 ; \cos'(0) = 0$$



Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

There exist $p_0, p_1, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_k(n)u_{n+k} = p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n$$

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

There exist $p_0, p_1, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_k(n)u_{n+k} = p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n$$

Examples:

- $n!$, Fibonacci numbers, harmonic numbers, Perrin numbers, Motzkin numbers, Catalan numbers, Apéry numbers, diagonal Delannoy numbers, Schröder numbers, ...

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

There exist $p_0, p_1, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_k(n)u_{n+k} = p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n$$

Examples:

- $n!$, Fibonacci numbers, harmonic numbers, Perrin numbers, Motzkin numbers, Catalan numbers, Apéry numbers, diagonal Delannoy numbers, Schröder numbers, ...
- Taylor coefficients of many, many functions. . .

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

There exist $p_0, p_1, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_k(n)u_{n+k} = p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n$$

Examples:

- $n!$, Fibonacci numbers, harmonic numbers, Perrin numbers, Motzkin numbers, Catalan numbers, Apéry numbers, diagonal Delannoy numbers, Schröder numbers, ...
- Taylor coefficients of many, many functions. . .
 - e.g. $\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \dots$

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

There exist $p_0, p_1, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_k(n)u_{n+k} = p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n$$

Examples:

- $n!$, Fibonacci numbers, harmonic numbers, Perrin numbers, Motzkin numbers, Catalan numbers, Apéry numbers, diagonal Delannoy numbers, Schröder numbers, ...
- Taylor coefficients of many, many functions. . .
 - e.g. $\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \dots$
sequence of coefficients: $\langle 0, 1, 0, -1/3!, 0, 1/5!, 0, -1/7!, \dots \rangle$

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

There exist $p_0, p_1, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_k(n)u_{n+k} = p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n$$

Examples:

- $n!$, Fibonacci numbers, harmonic numbers, Perrin numbers, Motzkin numbers, Catalan numbers, Apéry numbers, diagonal Delannoy numbers, Schröder numbers, ...
- Taylor coefficients of many, many functions. . .
 - e.g. $\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \dots$
sequence of coefficients: $\langle 0, 1, 0, -1/3!, 0, 1/5!, 0, -1/7!, \dots \rangle$
 $(n+2)(n+1)u_{n+2} = -u_n$

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients:

There exist $p_0, p_1, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_k(n)u_{n+k} = p_{k-1}(n)u_{n+k-1} + \dots + p_0(n)u_n$$

Examples:

- $n!$, Fibonacci numbers, harmonic numbers, Perrin numbers, Motzkin numbers, Catalan numbers, Apéry numbers, diagonal Delannoy numbers, Schröder numbers, ...
- Taylor coefficients of many, many functions. . .
 - e.g. $\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \dots$
sequence of coefficients: $\langle 0, 1, 0, -1/3!, 0, 1/5!, 0, -1/7!, \dots \rangle$
 $(n+2)(n+1)u_{n+2} = -u_n$
 $u_0 = 0$; $u_1 = 1$

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients

The OEIS Foundation is supported by donations from users of the OEIS and by a grant from the Simons Foundation.

0 1 3 6 2 7
: 13
: 20
23 : 12
10 22 11 21
OEIS
THE ON-LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

[The On-Line Encyclopedia of Integer Sequences® \(OEIS®\)](#)

Enter a sequence, word, or sequence number:

1,2,3,6,11,23,47,106,235

[Hints](#) [Welcome](#) [Video](#)

For more information about the Encyclopedia, see the [Welcome](#) page.

Languages: English [Shqip](#) [العربية](#) [Bosnia](#) [Euskara](#) [Català](#) [中文](#) (正體字, 简化字(1), 简化字(2))
[हिन्दी](#) [Հայերեն](#) [Dansk](#) [Nederlands](#) [Esperanto](#) [فارسی](#) [Suomi](#) [Français](#) [Deutsch](#) [Ελληνικά](#) [עברית](#)
[සිංහල](#) [Macedo](#) [Ido](#) [Bahasa Indonesia](#) [Italiano](#) [日本語](#) [ಕನ್ನಡ](#) [한국어](#) [Lietuvių](#) [മലയാളം](#) [Bokmal](#) [Norsk](#) [Polski](#) [Português](#)
[Română](#) [Русский](#) [Cymraeg](#) [Slovenščina](#) [Español](#) [Svenska](#) [Tamil](#) [தமிழ்](#) [Türkçe](#) [Українська](#) [اردو](#) [Tiếng Việt](#) [Cymraeg](#)

[Lookup](#) | [Welcome](#) | [Wiki](#) | [Register](#) | [Music](#) | [Plot 2](#) | [Demos](#) | [Index](#) | [Browse](#) | [More](#) | [WebCam](#)
[Contribute new seq. or comment](#) | [Format](#) | [Style Sheet](#) | [Transforms](#) | [Superseeker](#) | [Recent](#)
[The OEIS Community](#) | Maintained by [The OEIS Foundation Inc.](#)

[License Agreements](#) [Terms of Use](#) [Privacy Policy](#)...

Last modified October 7 17:45 EDT 2020. Contains 337622 sequences. (Running on oeis4.)

Approximately 25% of the sequences in the Online Encyclopedia of Integer Sequences are holonomic!

Manuel Kauers, ISSAC 2011

Holonomic Sequences

A sequence $\langle u_n \rangle_{n=0}^{\infty}$ of real number is **holonomic** (or **P-finite** or **P-recursive**) if it satisfies a linear recurrence relation with *polynomial* coefficients

Non-examples:

- 2^{2^n}
- $\log(n)$
- Bernoulli numbers
- prime numbers
- ...

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

- $\langle u_n + v_n \rangle_n$ — sum

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

- $\langle u_n + v_n \rangle_n$ — sum
- $\langle u_n v_n \rangle_n$ — (Hadamard) product

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

- $\langle u_n + v_n \rangle_n$ — sum
- $\langle u_n v_n \rangle_n$ — (Hadamard) product
- $\langle u_{n+1} \rangle_n$ — shift

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

- $\langle u_n + v_n \rangle_n$ — sum
- $\langle u_n v_n \rangle_n$ — (Hadamard) product
- $\langle u_{n+1} \rangle_n$ — shift
- $\langle \sum_{k=0}^n u_k \rangle_n$ — summation

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

- $\langle u_n + v_n \rangle_n$ — sum
- $\langle u_n v_n \rangle_n$ — (Hadamard) product
- $\langle u_{n+1} \rangle_n$ — shift
- $\langle \sum_{k=0}^n u_k \rangle_n$ — summation
- $\langle \sum_{k=0}^n u_k v_{n-k} \rangle_n$ — Cauchy product

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

- $\langle u_n + v_n \rangle_n$ — sum
- $\langle u_n v_n \rangle_n$ — (Hadamard) product
- $\langle u_{n+1} \rangle_n$ — shift
- $\langle \sum_{k=0}^n u_k \rangle_n$ — summation
- $\langle \sum_{k=0}^n u_k v_{n-k} \rangle_n$ — Cauchy product

*A great class for
computer algebra!*

Holonomic Sequences: Closure Properties

Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be holonomic sequences. Then so are:

- $\langle u_n + v_n \rangle_n$ — sum
- $\langle u_n v_n \rangle_n$ — (Hadamard) product
- $\langle u_{n+1} \rangle_n$ — shift
- $\langle \sum_{k=0}^n u_k \rangle_n$ — summation
- $\langle \sum_{k=0}^n u_k v_{n-k} \rangle_n$ — Cauchy product

*A great class for
computer algebra!*

All these operations are *effective* (computable)

Holonomic Sequences: Algorithmics

To **define** a holonomic sequence $\langle u_n \rangle_n$, you need only provide the recurrence relation together with **finitely** many initial values

$u_0, \dots, u_r$

Holonomic Sequences: Algorithmics

To **define** a holonomic sequence $\langle u_n \rangle_n$, you need only provide the recurrence relation together with **finitely** many initial values $u_0, \dots, u_r$

Theorem

For two holonomic sequences $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ with initial values in $\overline{Q}$, the equality problem “ $u_n \equiv v_n$?” is decidable.

Holonomic Sequences: Algorithmics

To **define** a holonomic sequence $\langle u_n \rangle_n$, you need only provide the recurrence relation together with **finitely** many initial values $u_0, \dots, u_r$

Theorem

For two holonomic sequences $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ with initial values in $\overline{\mathbb{Q}}$, the equality problem “ $u_n \equiv v_n$?” is decidable.

“Equality (identity) is decidable for holonomic sequences”

Holonomic Functions

A function $f : I \rightarrow \mathbb{R}$ is **holonomic** (or **D-finite**) if it satisfies a linear differential equation with *polynomial* coefficients:

Holonomic Functions

A function $f : I \rightarrow \mathbb{R}$ is **holonomic** (or **D-finite**) if it satisfies a linear differential equation with *polynomial* coefficients:

There exist $p_0, p_1, p_2, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \dots + p_k(x)f^{(k)}(x) = 0$$

Holonomic Functions

A function $f : I \rightarrow \mathbb{R}$ is **holonomic** (or **D-finite**) if it satisfies a linear differential equation with *polynomial* coefficients:

There exist $p_0, p_1, p_2, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \dots + p_k(x)f^{(k)}(x) = 0$$

Examples:

- algebraic functions, $\sin(x)$, e^x , $\log x$, Bessel functions, hypergeometric functions, Hankel functions, Airy functions, polylogarithms, elliptic integrals, ...

Holonomic Functions

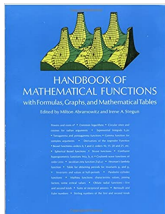
A function $f : I \rightarrow \mathbb{R}$ is **holonomic** (or **D-finite**) if it satisfies a linear differential equation with *polynomial* coefficients:

There exist $p_0, p_1, p_2, \dots, p_k \in \mathbb{Q}[x]$ such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \dots + p_k(x)f^{(k)}(x) = 0$$

Examples:

- algebraic functions, $\sin(x)$, e^x , $\log x$, Bessel functions, hypergeometric functions, Hankel functions, Airy functions, polylogarithms, elliptic integrals, ...



Approximately 60% of the functions in Abramowitz and Stegun's handbook are holonomic!

Holonomic Functions: Closure Properties

Let $f(x)$ and $g(x)$ be holonomic functions. Then so are:

Holonomic Functions: Closure Properties

Let $f(x)$ and $g(x)$ be holonomic functions. Then so are:

- $f(x) + g(x)$ — sum

Holonomic Functions: Closure Properties

Let $f(x)$ and $g(x)$ be holonomic functions. Then so are:

- $f(x) + g(x)$ — sum
- $f(x)g(x)$ — product

Holonomic Functions: Closure Properties

Let $f(x)$ and $g(x)$ be holonomic functions. Then so are:

- $f(x) + g(x)$ — sum
- $f(x)g(x)$ — product
- $\int_0^x f(t) dt$ — indefinite integration

Holonomic Functions: Closure Properties

Let $f(x)$ and $g(x)$ be holonomic functions. Then so are:

- $f(x) + g(x)$ — sum
- $f(x)g(x)$ — product
- $\int_0^x f(t) dt$ — indefinite integration
- $f'(x)$ — differentiation

Holonomic Functions: Closure Properties

Let $f(x)$ and $g(x)$ be holonomic functions. Then so are:

- $f(x) + g(x)$ — sum
- $f(x)g(x)$ — product
- $\int_0^x f(t) dt$ — indefinite integration
- $f'(x)$ — differentiation
- $f(a(x))$ — right composition by algebraic function

Holonomic Functions: Closure Properties

Let $f(x)$ and $g(x)$ be holonomic functions. Then so are:

- $f(x) + g(x)$ — sum
- $f(x)g(x)$ — product
- $\int_0^x f(t) dt$ — indefinite integration
- $f'(x)$ — differentiation
- $f(a(x))$ — right composition by algebraic function

All these operations are *effective*

Holonomic Functions: Algorithmics

Let $I \subseteq \mathbb{R}$ be an interval, and assume that $0 \in I$. To **define** a holonomic function $f : I \rightarrow \mathbb{R}$, you need only provide the differential equation together with **finitely** many initial values $f(0), f'(0), f''(0), \dots, f^{(r)}(0)$ (provided that 0 is not a root of the polynomial coefficient of the highest-order term of the differential equation).

Holonomic Functions: Algorithmics

Let $I \subseteq \mathbb{R}$ be an interval, and assume that $0 \in I$. To **define** a holonomic function $f : I \rightarrow \mathbb{R}$, you need only provide the differential equation together with **finitely** many initial values $f(0), f'(0), f''(0), \dots, f^{(r)}(0)$ (provided that 0 is not a root of the polynomial coefficient of the highest-order term of the differential equation).

Theorem

For two holonomic functions $f, g : I \rightarrow \mathbb{R}$ with initial values in $\overline{\mathbb{Q}}$, the equality problem “ $f \equiv g$?” is decidable.

Holonomic Functions: Algorithmics

Let $I \subseteq \mathbb{R}$ be an interval, and assume that $0 \in I$. To **define** a holonomic function $f : I \rightarrow \mathbb{R}$, you need only provide the differential equation together with **finitely** many initial values $f(0), f'(0), f''(0), \dots, f^{(r)}(0)$ (provided that 0 is not a root of the polynomial coefficient of the highest-order term of the differential equation).

Theorem

For two holonomic functions $f, g : I \rightarrow \mathbb{R}$ with initial values in $\overline{\mathbb{Q}}$, the equality problem “ $f \equiv g$?” is decidable.

“Equality (identity) is decidable for holonomic functions”

The 'Fundamental Theorem' of Holonomy Theory

Theorem

Let $\langle u_n \rangle_{n=0}^{\infty}$ be a holonomic sequence.

Then the generating function $f(x) = \sum_{n=0}^{\infty} u_n x^n$ is holonomic.

The 'Fundamental Theorem' of Holonomy Theory

Theorem

Let $\langle u_n \rangle_{n=0}^{\infty}$ be a holonomic sequence.

Then the generating function $f(x) = \sum_{n=0}^{\infty} u_n x^n$ is holonomic.

Conversely, the sequence of coefficients of the Taylor expansion of a holonomic function is always holonomic.

The 'Fundamental Theorem' of Holonomy Theory

Theorem

Let $\langle u_n \rangle_{n=0}^{\infty}$ be a holonomic sequence.

Then the generating function $f(x) = \sum_{n=0}^{\infty} u_n x^n$ is holonomic.

Conversely, the sequence of coefficients of the Taylor expansion of a holonomic function is always holonomic.

These transformations are effective.

The 'Fundamental Theorem' of Holonomy Theory

Theorem

Let $\langle u_n \rangle_{n=0}^{\infty}$ be a holonomic sequence.

Then the generating function $f(x) = \sum_{n=0}^{\infty} u_n x^n$ is holonomic.

Conversely, the sequence of coefficients of the Taylor expansion of a holonomic function is always holonomic.

These transformations are effective.

Example — the Fibonacci sequence $\langle F_n \rangle_n$:

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

Holonomic Techniques: A Brief History

Consider the recurrence $bu_{n+1} = au_n$

Holonomic Techniques: A Brief History

Consider the recurrence $bu_{n+1} = au_n$

Then $u_n = (a/b)^n u_0$: a **geometric** sequence

Holonomic Techniques: A Brief History

Consider the recurrence $bu_{n+1} = au_n$

Then $u_n = (a/b)^n u_0$: a **geometric** sequence

Now replace the constants a and b by polynomials:

$$B(n)u_{n+1} = A(n)u_n$$

Holonomic Techniques: A Brief History

Consider the recurrence $bu_{n+1} = au_n$

Then $u_n = (a/b)^n u_0$: a **geometric** sequence

Now replace the constants a and b by polynomials:

$$B(n)u_{n+1} = A(n)u_n$$

Such recurrences define **hypergeometric sequences**

Holonomic Techniques: A Brief History

Consider the recurrence $bu_{n+1} = au_n$

Then $u_n = (a/b)^n u_0$: a **geometric** sequence

Now replace the constants a and b by polynomials:

$$B(n)u_{n+1} = A(n)u_n$$

Such recurrences define **hypergeometric sequences**

The corresponding power series $\sum_{n=0}^{\infty} u_n x^n = {}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x \right]$
are **hypergeometric functions**

Holonomic Techniques: A Brief History

Consider the recurrence $bu_{n+1} = au_n$

Then $u_n = (a/b)^n u_0$: a **geometric** sequence

Now replace the constants a and b by polynomials:

$$B(n)u_{n+1} = A(n)u_n$$

Such recurrences define **hypergeometric sequences**

The corresponding power series $\sum_{n=0}^{\infty} u_n x^n = {}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x \right]$
are **hypergeometric functions**

These include exponential, logarithmic, trigonometric, binomial, and Bessel functions, as well as classical orthogonal polynomial sequences of Legendre, Chebyshev, Laguerre, Hermite, ...

Holonomic Techniques: A Brief History

Consider the recurrence $bu_{n+1} = au_n$

Then $u_n = (a/b)^n u_0$: a **geometric** sequence

Now replace the constants a and b by polynomials:

$$B(n)u_{n+1} = A(n)u_n$$

Such recurrences define **hypergeometric sequences**

The corresponding power series $\sum_{n=0}^{\infty} u_n x^n = {}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x \right]$
are **hypergeometric functions**

- The study of hypergeometric functions originated with Wallis; there followed extensive work by Euler, Gauss, Kummer, Riemann, Ramanujan, . . . , Beukers, Zudilin . . .
- Nowadays a major research area in its own right

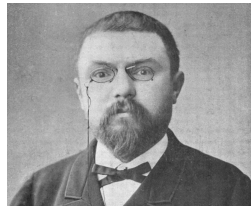
Holonomic Techniques: A Brief History

Hypergeometric sequences are **first-order** holonomic sequences

Holonomic Techniques: A Brief History

Hypergeometric sequences are **first-order** holonomic sequences

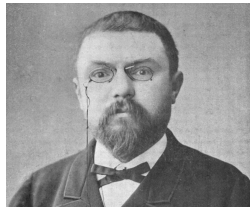
The systematic study of holonomic sequences of arbitrary order began with Poincaré in the late 19th century, focusing on *asymptotics*



Holonomic Techniques: A Brief History

Hypergeometric sequences are **first-order** holonomic sequences

The systematic study of holonomic sequences of arbitrary order began with Poincaré in the late 19th century, focusing on *asymptotics*



Other contributors to asymptotics (and more!) include Perron, Kooman, Salvy, Flajolet ...



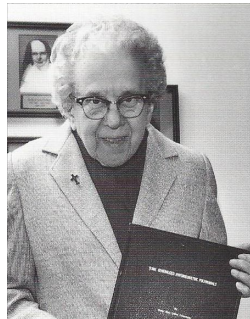
Holonomic Techniques: A Brief History

The [algorithmic](#) approach to holonomic sequences

Holonomic Techniques: A Brief History

The **algorithmic** approach to holonomic sequences

A paradigm shift with the advent of Sister Mary Celine Fassenmyer's PhD thesis (1945, University of Michigan)

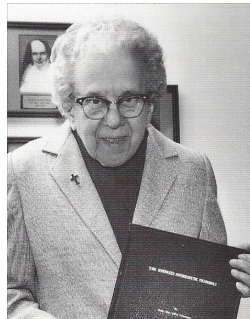


Holonomic Techniques: A Brief History

The **algorithmic** approach to holonomic sequences

A paradigm shift with the advent of Sister Mary Celine Fasenmyer's PhD thesis (1945, University of Michigan)

Fasenmyer was a Catholic nun ("Sister Celine") from Pennsylvania.



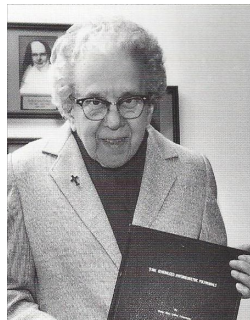
Holonomic Techniques: A Brief History

The **algorithmic** approach to holonomic sequences

A paradigm shift with the advent of Sister Mary Celine Fasenmyer's PhD thesis (1945, University of Michigan)

Fasenmyer was a Catholic nun ("Sister Celine") from Pennsylvania.

She developed an algorithmic method to find recurrence relations satisfied by sums of terms of hypergeometric sequences, now known as "Sister Celine's method"



Holonomic Techniques: A Brief History

The **algorithmic** approach to holonomic sequences

- More work in the 70s and 80s by Gosper, Stanley, Gessel, Lipshitz, ...

Holonomic Techniques: A Brief History

The **algorithmic** approach to holonomic sequences

- More work in the 70s and 80s by Gosper, Stanley, Gessel, Lipshitz, . . .
- Weaponised by Zeilberger in his seminal 1990 paper:

Journal of Computational and Applied Mathematics 32 (1990) 321–368
North-Holland

321

A holonomic systems approach to special
functions identities *

Doron ZEILBERGER

Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Holonomic Techniques: A Brief History

The **algorithmic** approach to holonomic sequences

- More work in the 70s and 80s by Gosper, Stanley, Gessel, Lipshitz, . . .
- Weaponised by Zeilberger in his seminal 1990 paper:

Journal of Computational and Applied Mathematics 32 (1990) 321–368
North-Holland

321

A holonomic systems approach to special functions identities *

Doron ZEILBERGER

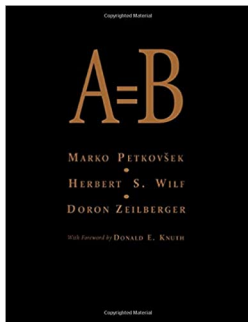
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

- ⇒ In that paper, Zeilberger elaborated the theory of **multivariable** holonomic systems, based on Bernstein's deep theory of D-modules

Holonomic Techniques: A Brief History

The **algorithmic** approach to holonomic sequences

From the Foreword:



"I fell in love with these procedures as soon as I learned them, because they worked for me immediately. Not only did they dispose of sums that I had wrestled with long and hard in the past, they also knocked off two new problems that I was working on at the time I first tried them. The success rate was astonishing.

[...]

This book will help you reach new frontiers."

Donald Knuth, 1995

Holonomic Techniques: Some Limitations

- Deciding inequalities
- Computing asymptotics

Holonomic Techniques: Some Limitations

- Deciding inequalities
- Computing asymptotics

Problem (Hypergeometric Inequality)

*Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be two hypergeometric sequences.
Is it the case that, for all n , $u_n \geq v_n$?*

Holonomic Techniques: Some Limitations

- Deciding inequalities
- Computing asymptotics

Problem (Hypergeometric Inequality)

*Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be two hypergeometric sequences.
Is it the case that, for all n , $u_n \geq v_n$?*

Problem (Positivity)

*Let $\langle u_n \rangle_n$ be a holonomic sequence.
Is it the case that, for all n , $u_n \geq 0$?*

Holonomic Techniques: Some Limitations

- Deciding inequalities
- Computing asymptotics

Problem (Hypergeometric Inequality)

*Let $\langle u_n \rangle_n$ and $\langle v_n \rangle_n$ be two hypergeometric sequences.
Is it the case that, for all n , $u_n \geq v_n$?*

Problem (Positivity)

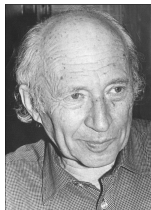
*Let $\langle u_n \rangle_n$ be a holonomic sequence.
Is it the case that, for all n , $u_n \geq 0$?*

Problem (Minimality)

*Let $\langle u_n \rangle_n$ be a holonomic sequence.
Is $\langle u_n \rangle_n$ minimal ?
Special case: does $u_n \rightarrow 0$ as $n \rightarrow \infty$?*

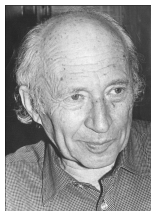
David Madore Anecdote: On Algebra vs. Analysis

I was once dining at the IHÉS (it must have been in '96) and I happened to be seated right next to I. M. Gel'fand. When I said I wanted to become a mathematician, he asked whether I preferred Algebra or Analysis, and I said Algebra.



David Madore Anecdote: On Algebra vs. Analysis

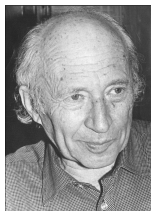
I was once dining at the IHÉS (it must have been in '96) and I happened to be seated right next to I. M. Gel'fand. When I said I wanted to become a mathematician, he asked whether I preferred Algebra or Analysis, and I said Algebra.



"I don't understand why people want to study Algebra," replied Gel'fand: "because in Algebra you write $a=b$, but if $a=b$ then a and b are the same thing, so you're writing $a=a$ and what's the point of that?"

David Madore Anecdote: On Algebra vs. Analysis

I was once dining at the IHÉS (it must have been in '96) and I happened to be seated right next to I. M. Gel'fand. When I said I wanted to become a mathematician, he asked whether I preferred Algebra or Analysis, and I said Algebra.



"I don't understand why people want to study Algebra," replied Gel'fand: "because in Algebra you write $a=b$, but if $a=b$ then a and b are the same thing, so you're writing $a=a$ and what's the point of that?"

Well, I'm not as smart as Gel'fand so I wrote my Ph.D. in Algebraic Geometry anyway.

David Madore, 2013

Asymptotics and Minimal Solutions

Consider the Fibonacci recurrence equation: $x_{n+2} = x_{n+1} + x_n$

Asymptotics and Minimal Solutions

Consider the Fibonacci recurrence equation: $x_{n+2} = x_{n+1} + x_n$

- Any sequence $\langle u_n \rangle_{n=0}^{\infty}$ satisfying this equation is a *solution* of the recurrence

Asymptotics and Minimal Solutions

Consider the Fibonacci recurrence equation: $x_{n+2} = x_{n+1} + x_n$

- Any sequence $\langle u_n \rangle_{n=0}^{\infty}$ satisfying this equation is a *solution* of the recurrence
- The set of solutions is a two-dimensional vector space

Asymptotics and Minimal Solutions

Consider the Fibonacci recurrence equation: $x_{n+2} = x_{n+1} + x_n$

- Any sequence $\langle u_n \rangle_{n=0}^{\infty}$ satisfying this equation is a *solution* of the recurrence
 - The set of solutions is a two-dimensional vector space
- $\Rightarrow$ A (non-zero) solution $\langle u_n \rangle_n$ is *minimal* if, for any other linearly independent solution $\langle v_n \rangle_n$, we have $u_n/v_n \rightarrow 0$ as $n \rightarrow \infty$

Asymptotics and Minimal Solutions

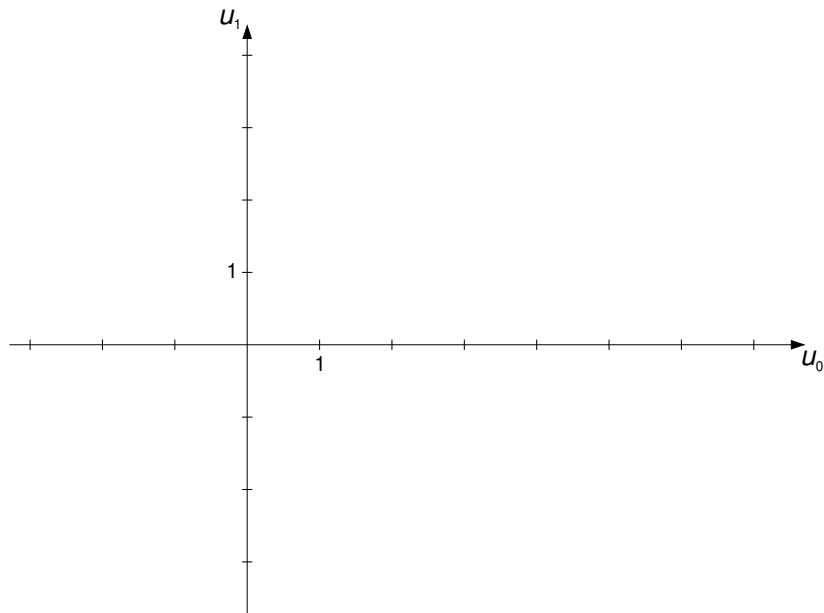
Consider the Fibonacci recurrence equation: $x_{n+2} = x_{n+1} + x_n$

- Any sequence $\langle u_n \rangle_{n=0}^{\infty}$ satisfying this equation is a *solution* of the recurrence
 - The set of solutions is a two-dimensional vector space
- ⇒ A (non-zero) solution $\langle u_n \rangle_n$ is *minimal* if, for any other linearly independent solution $\langle v_n \rangle_n$, we have $u_n/v_n \rightarrow 0$ as $n \rightarrow \infty$
- Any solution $\langle u_n \rangle_n$ to the Fibonacci recurrence can be written

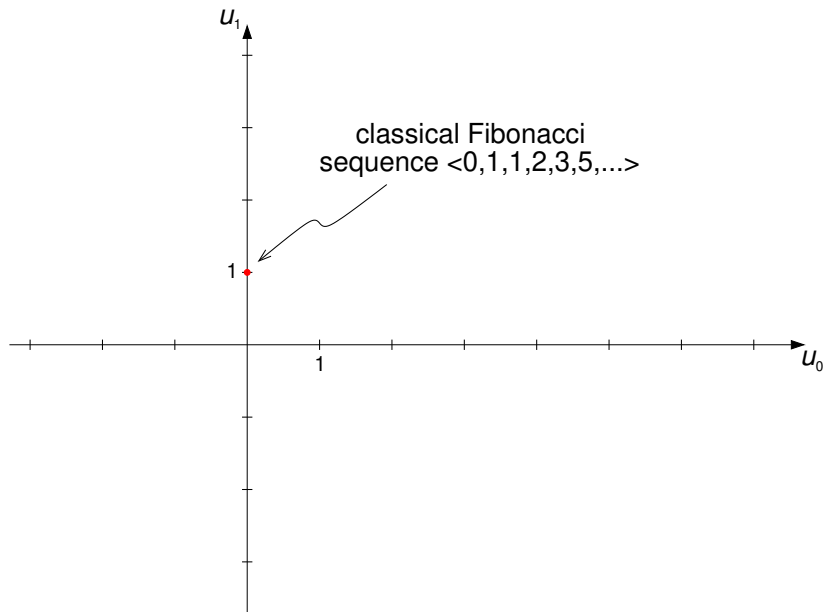
$$u_n = a\phi^n + b\bar{\phi}^n$$

for some $a, b \in \mathbb{R}$, with $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$

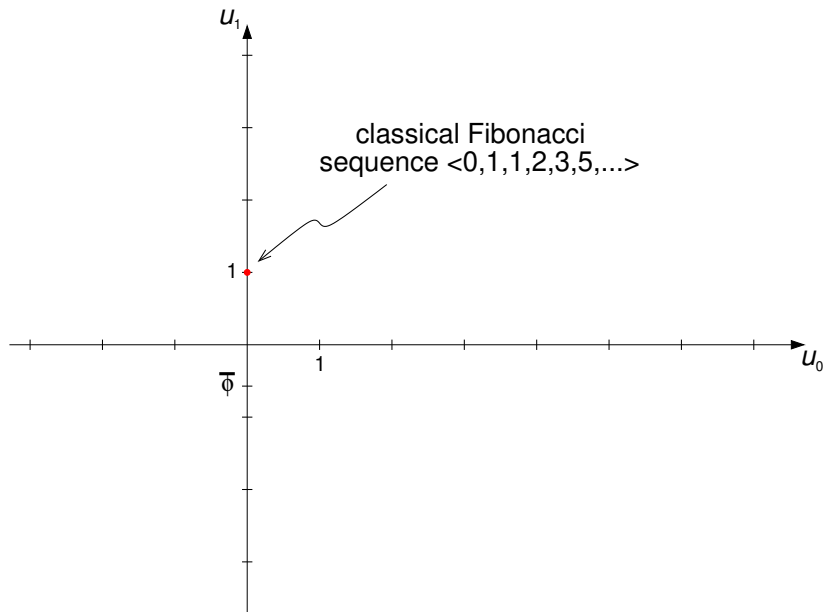
The Vector Space of Fibonacci Sequences



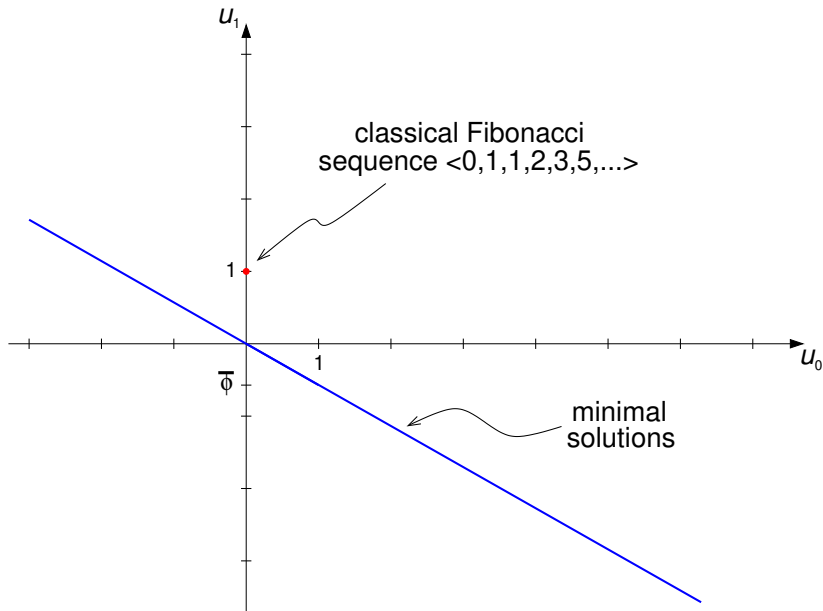
The Vector Space of Fibonacci Sequences



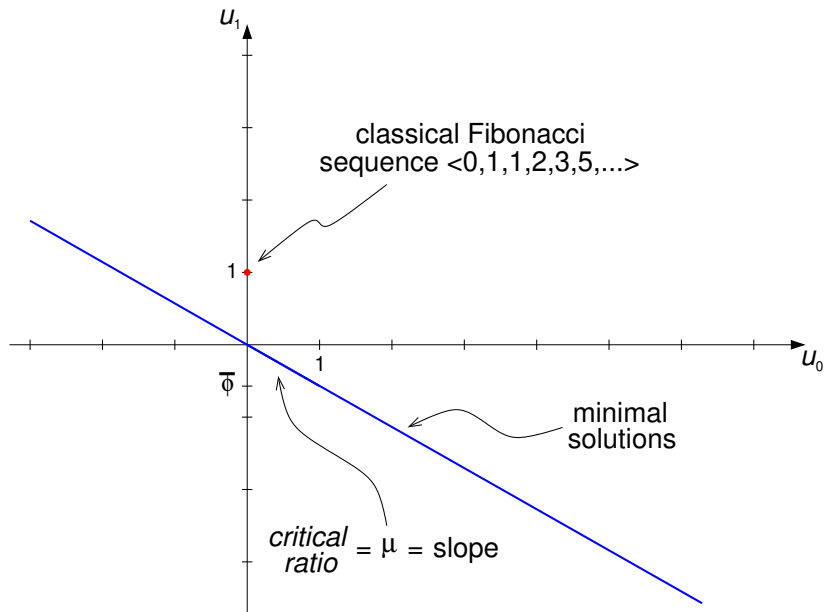
The Vector Space of Fibonacci Sequences



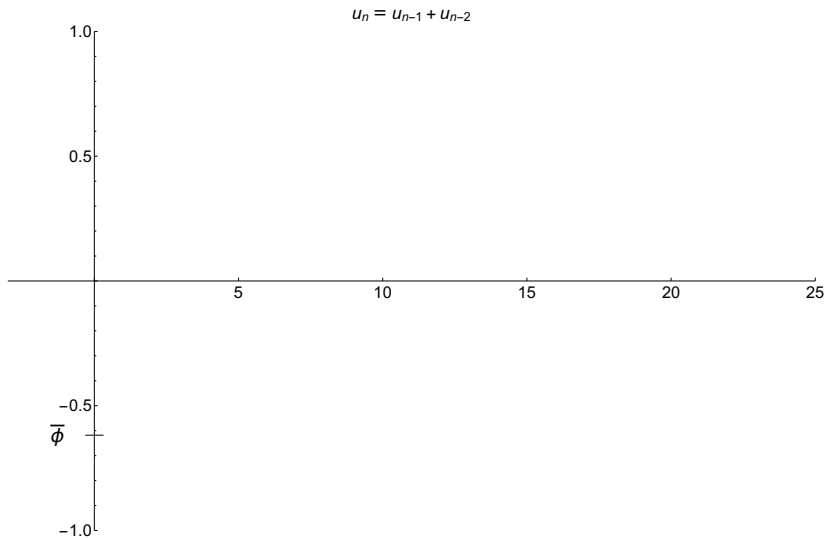
The Vector Space of Fibonacci Sequences



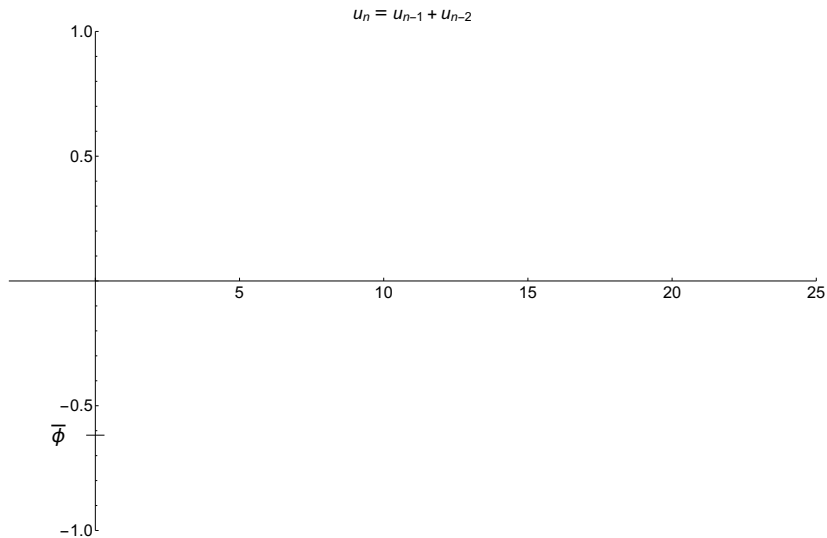
The Vector Space of Fibonacci Sequences



Plotting u_n against n

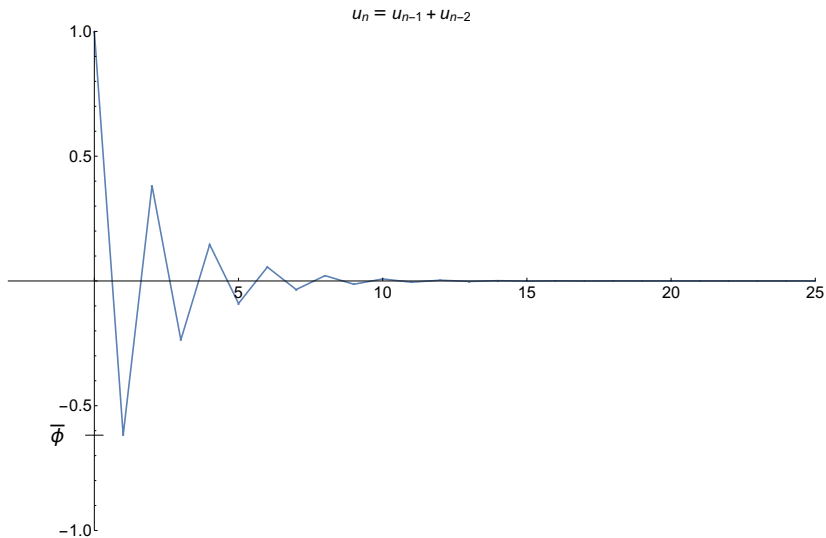


Plotting u_n against n



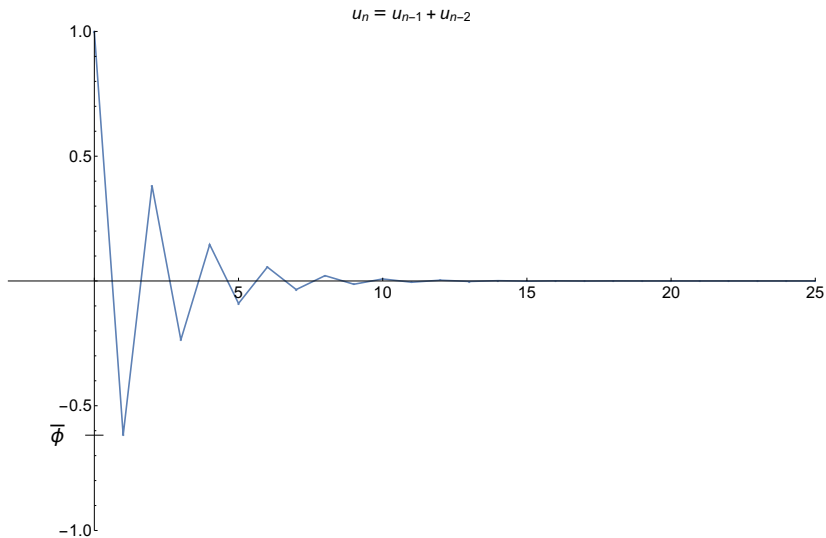
$$(u_0, u_1) = (1, \bar{\phi})$$

Plotting u_n against n



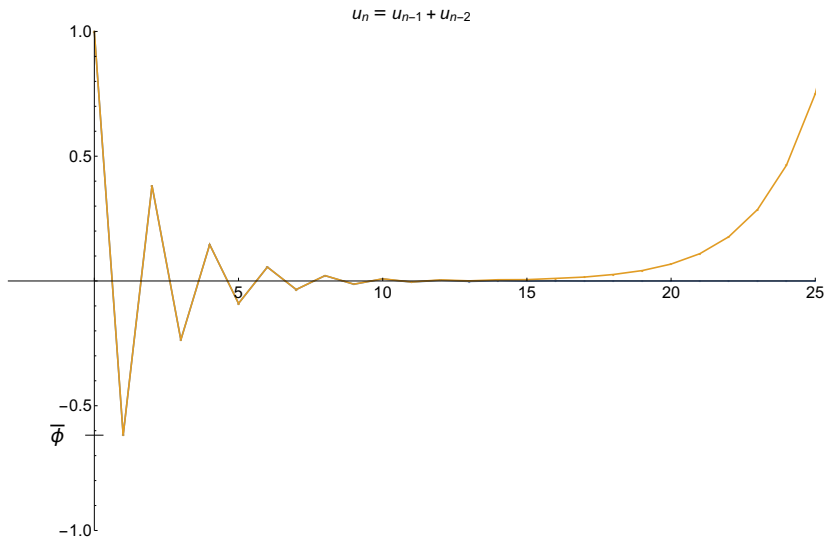
$$(u_0, u_1) = (1, \bar{\phi})$$

Plotting u_n against n



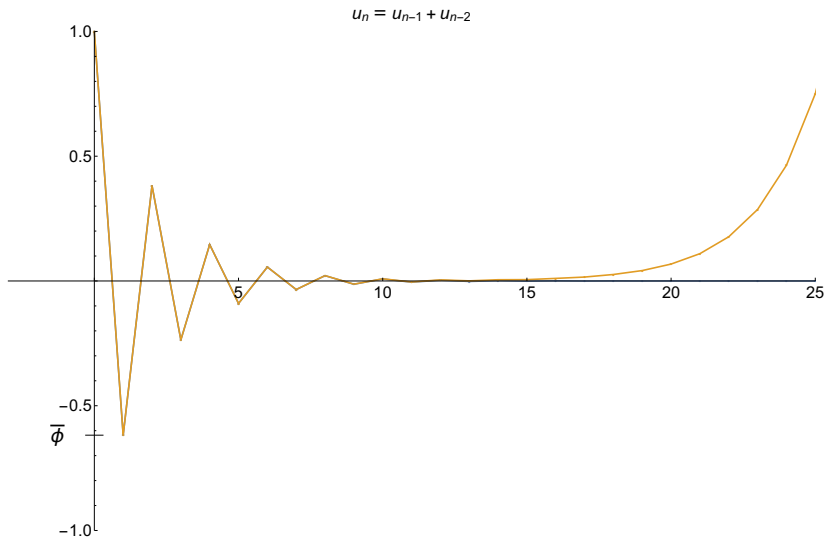
$$(u_0, u_1) = (1, \bar{\phi}) // (1, \bar{\phi} + 0.00001)$$

Plotting u_n against n



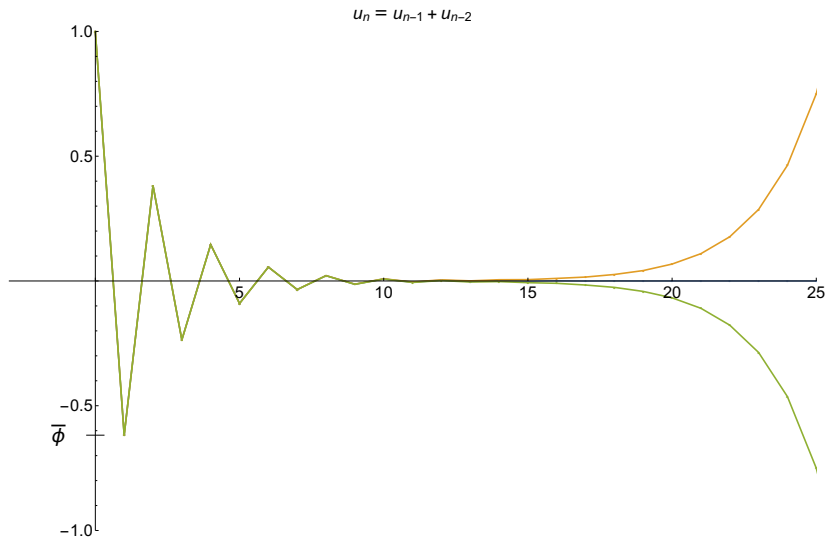
$$(u_0, u_1) = (1, \bar{\phi}) // (1, \bar{\phi} + 0.00001)$$

Plotting u_n against n



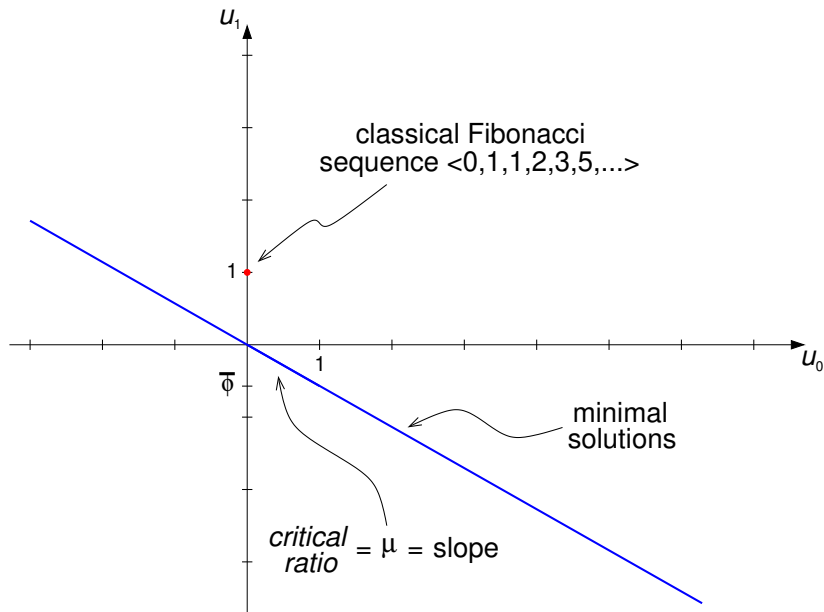
$$(u_0, u_1) = (1, \bar{\phi}) // (1, \bar{\phi} + 0.00001) // (1, \bar{\phi} - 0.00001)$$

Plotting u_n against n



$$(u_0, u_1) = (1, \bar{\phi}) // (1, \bar{\phi} + 0.00001) // (1, \bar{\phi} - 0.00001)$$

The Vector Space of Fibonacci Sequences



Apéry Numbers Again!

Recall the Apéry recurrence equation:

$$(n+2)^3 x_{n+2} - (2n+3)(17n^2 + 51n + 39)x_{n+1} + (n+1)^3 x_n = 0$$

Apéry Numbers Again!

Recall the Apéry recurrence equation:

$$(n+2)^3 x_{n+2} - (2n+3)(17n^2 + 51n + 39)x_{n+1} + (n+1)^3 x_n = 0$$

- The Apéry numbers $\langle A_n \rangle_{n=0}^\infty = \langle 1, 5, 73, 1445, 33001, \dots \rangle$ are a solution of this recurrence

Apéry Numbers Again!

Recall the Apéry recurrence equation:

$$(n+2)^3 x_{n+2} - (2n+3)(17n^2 + 51n + 39)x_{n+1} + (n+1)^3 x_n = 0$$

- The Apéry numbers $\langle A_n \rangle_{n=0}^\infty = \langle 1, 5, 73, 1445, 33001, \dots \rangle$ are a solution of this recurrence
- Another solution is $\langle B_n \rangle_{n=0}^\infty = \langle 0, 6, 351/4, 62531/36, \dots \rangle$

Apéry Numbers Again!

Recall the Apéry recurrence equation:

$$(n+2)^3 x_{n+2} - (2n+3)(17n^2 + 51n + 39)x_{n+1} + (n+1)^3 x_n = 0$$

- The Apéry numbers $\langle A_n \rangle_{n=0}^\infty = \langle 1, 5, 73, 1445, 33001, \dots \rangle$ are a solution of this recurrence
- Another solution is $\langle B_n \rangle_{n=0}^\infty = \langle 0, 6, 351/4, 62531/36, \dots \rangle$
- Letting $u_n = B_n - \zeta(3)A_n$, by linearity $\langle u_n \rangle_{n=0}^\infty$ is also a solution of the Apéry recurrence

Apéry Numbers Again!

Recall the Apéry recurrence equation:

$$(n+2)^3 x_{n+2} - (2n+3)(17n^2 + 51n + 39)x_{n+1} + (n+1)^3 x_n = 0$$

- The Apéry numbers $\langle A_n \rangle_{n=0}^\infty = \langle 1, 5, 73, 1445, 33001, \dots \rangle$ are a solution of this recurrence
- Another solution is $\langle B_n \rangle_{n=0}^\infty = \langle 0, 6, 351/4, 62531/36, \dots \rangle$
- Letting $u_n = B_n - \zeta(3)A_n$, by linearity $\langle u_n \rangle_{n=0}^\infty$ is also a solution of the Apéry recurrence
- The punchline of Apéry's proof is the fact that $u_n \rightarrow 0$ fast as $n \rightarrow \infty$, showing that $\zeta(3)$ cannot be rational

Apéry Numbers Again!

Recall the Apéry recurrence equation:

$$(n+2)^3 x_{n+2} - (2n+3)(17n^2 + 51n + 39)x_{n+1} + (n+1)^3 x_n = 0$$

- The Apéry numbers $\langle A_n \rangle_{n=0}^\infty = \langle 1, 5, 73, 1445, 33001, \dots \rangle$ are a solution of this recurrence
- Another solution is $\langle B_n \rangle_{n=0}^\infty = \langle 0, 6, 351/4, 62531/36, \dots \rangle$
- Letting $u_n = B_n - \zeta(3)A_n$, by linearity $\langle u_n \rangle_{n=0}^\infty$ is also a solution of the Apéry recurrence
- The punchline of Apéry's proof is the fact that $u_n \rightarrow 0$ fast as $n \rightarrow \infty$, showing that $\zeta(3)$ cannot be rational
- An immediate consequence is that $\langle u_n \rangle_{n=0}^\infty$ is a minimal solution, and the critical ratio of the Apéry recurrence is therefore $\mu = 5 - 6/\zeta(3)$

Comparing Real Numbers

$$\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-(n^2/10^{10})} \stackrel{?}{=} \sqrt{\pi}$$

Comparing Real Numbers

$$\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-(n^2/10^{10})} \stackrel{?}{=} \sqrt{\pi}$$

FALSE! **But** the two sides agree to over *42 billion digits!*

Comparing Real Numbers

$$\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-(n^2/10^{10})} \stackrel{?}{=} \sqrt{\pi}$$

FALSE! **But** the two sides agree to over *42 billion digits!*

Strange Series and High Precision Fraud

J. M. Borwein and P. B. Borwein



Periods

Definition (Kontsevich & Zagier, 2001)

A **period** is a complex number whose real and imaginary parts are values of integrals of algebraic functions over semialgebraic domains.



Periods

Definition (Kontsevich & Zagier, 2001)

A **period** is a complex number whose real and imaginary parts are values of integrals of algebraic functions over semialgebraic domains.

$$\int_D g(x_1, \dots, x_k) dx_1 \dots dx_k$$

where $D \subseteq \mathbb{R}^k$ is a semialgebraic set and $g : D \rightarrow \mathbb{R}$ is an algebraic function



Periods

Definition (Kontsevich & Zagier, 2001)

A **period** is a complex number whose real and imaginary parts are values of integrals of algebraic functions over semialgebraic domains.

$$\int_D g(x_1, \dots, x_k) dx_1 \dots dx_k$$

where $D \subseteq \mathbb{R}^k$ is a semialgebraic set and $g : D \rightarrow \mathbb{R}$ is an algebraic function

$$\sqrt[3]{7} = \int_0^{\sqrt[3]{7}} dx$$



Periods

Definition (Kontsevich & Zagier, 2001)

A **period** is a complex number whose real and imaginary parts are values of integrals of algebraic functions over semialgebraic domains.

$$\int_D g(x_1, \dots, x_k) dx_1 \dots dx_k$$

where $D \subseteq \mathbb{R}^k$ is a semialgebraic set and $g : D \rightarrow \mathbb{R}$ is an algebraic function

$$\sqrt[3]{7} = \int_0^{\sqrt[3]{7}} dx$$

$$\pi = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$



Periods

Definition (Kontsevich & Zagier, 2001)

A **period** is a complex number whose real and imaginary parts are values of integrals of algebraic functions over semialgebraic domains.

$$\int_D g(x_1, \dots, x_k) dx_1 \dots dx_k$$

where $D \subseteq \mathbb{R}^k$ is a semialgebraic set and $g : D \rightarrow \mathbb{R}$ is an algebraic function

$$\sqrt[3]{7} = \int_0^{\sqrt[3]{7}} dx$$

$$\pi = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

$$\log 2 = \int_1^2 \frac{dx}{x}$$



Periods

Definition (Kontsevich & Zagier, 2001)

A **period** is a complex number whose real and imaginary parts are values of integrals of algebraic functions over semialgebraic domains.



$$\int_D g(x_1, \dots, x_k) dx_1 \dots dx_k$$

where $D \subseteq \mathbb{R}^k$ is a semialgebraic set and $g : D \rightarrow \mathbb{R}$ is an algebraic function

$$\sqrt[3]{7} = \int_0^{\sqrt[3]{7}} dx$$

$$\pi = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

$$\log 2 = \int_1^2 \frac{dx}{x}$$

$$\zeta(3) = \iiint_{0 < x < y < z < 1} \frac{dx dy dz}{(1-x)yz}$$

Periods

In fact, the notion of period goes back to Grothendieck

Periods

In fact, the notion of period goes back to Grothendieck

Conjecture (Grothendieck Period Conjecture)

Let X be a smooth projective variety over $\overline{\mathbb{Q}}$ and let $X_{\mathbb{C}}^{\text{an}}$ denote the compact complex analytic manifold that it defines. The Grothendieck Period Conjecture in codimension k on X asserts that any class α in the algebraic de Rham cohomology group $H_{\text{dR}}^{2k}(X/\overline{\mathbb{Q}})$ of X over $\overline{\mathbb{Q}}$ such that

$$\frac{1}{(2\pi i)^k} \int_{\gamma} \alpha \in \mathbb{Q}$$

for every rational homology class γ in $H_{2k}(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$ is the class in algebraic de Rham cohomology of some algebraic cycle of codimension k in X , with rational coefficients.



Thankfully, an essentially equivalent conjecture was formulated by Kontsevich and Zagier:

Thankfully, an essentially equivalent conjecture was formulated by Kontsevich and Zagier:

Conjecture (Kontsevich & Zagier, 2001)

It is decidable whether two given periods are equal or not.

Thankfully, an essentially equivalent conjecture was formulated by Kontsevich and Zagier:

Conjecture (Kontsevich & Zagier, 2001)

It is decidable whether two given periods are equal or not.

This remains wide open. But in the meantime, following the breakthrough work of Huber & Wüstholz (*Transcendence and Linear Relations of 1-Periods*, 2018–2022), we have:

Thankfully, an essentially equivalent conjecture was formulated by Kontsevich and Zagier:

Conjecture (Kontsevich & Zagier, 2001)

It is decidable whether two given periods are equal or not.

This remains wide open. But in the meantime, following the breakthrough work of Huber & Wüstholz (*Transcendence and Linear Relations of 1-Periods*, 2018–2022), we have:

Theorem (Sertöz, O., Worrell, 2025 [currently under review])

Equality of 1-periods is decidable. In particular, given a 1-period α , there is an algorithm which can determine whether $\alpha = 0$ or not.

Positivity and Minimality: A Simple Example

Consider the second-order, degree-1 recurrence

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

Positivity and Minimality: A Simple Example

Consider the second-order, degree-1 recurrence

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

Problem

*Given $u_{-1}, u_0 \in \mathbb{Q}$ (or $\overline{\mathbb{Q}} \cap \mathbb{R}$), we want to decide whether the corresponding full sequence $\langle u_n \rangle_{n=-1}^{\infty}$ is **minimal** and/or **positive**.*

Positivity and Minimality: A Simple Example

Consider the second-order, degree-1 recurrence

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

Problem

*Given $u_{-1}, u_0 \in \mathbb{Q}$ (or $\overline{\mathbb{Q}} \cap \mathbb{R}$), we want to decide whether the corresponding full sequence $\langle u_n \rangle_{n=-1}^{\infty}$ is **minimal** and/or **positive**.*

- For each n , consider the characteristic polynomial $(2n + 3)z^2 - (6n + 7)z + (4n + 1)$ and denote the roots λ_n and Λ_n

Positivity and Minimality: A Simple Example

Consider the second-order, degree-1 recurrence

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

Problem

*Given $u_{-1}, u_0 \in \mathbb{Q}$ (or $\overline{\mathbb{Q}} \cap \mathbb{R}$), we want to decide whether the corresponding full sequence $\langle u_n \rangle_{n=-1}^{\infty}$ is **minimal** and/or **positive**.*

- For each n , consider the characteristic polynomial $(2n + 3)z^2 - (6n + 7)z + (4n + 1)$ and denote the roots λ_n and Λ_n
- Observe that $\lambda_n \rightarrow \lambda = 1$ and $\Lambda_n \rightarrow \Lambda = 2$ as $n \rightarrow \infty$

Positivity and Minimality: A Simple Example

Consider the second-order, degree-1 recurrence

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

Problem

*Given $u_{-1}, u_0 \in \mathbb{Q}$ (or $\overline{\mathbb{Q}} \cap \mathbb{R}$), we want to decide whether the corresponding full sequence $\langle u_n \rangle_{n=-1}^\infty$ is **minimal** and/or **positive**.*

- For each n , consider the characteristic polynomial $(2n + 3)z^2 - (6n + 7)z + (4n + 1)$ and denote the roots λ_n and Λ_n
- Observe that $\lambda_n \rightarrow \lambda = 1$ and $\Lambda_n \rightarrow \Lambda = 2$ as $n \rightarrow \infty$
- By the Poincaré-Perron theorem, for any solution sequence $\langle u_n \rangle_n$, we have either $u_n/u_{n-1} \rightarrow \lambda$ or $u_n/u_{n-1} \rightarrow \Lambda$

Positivity and Minimality: A Simple Example

Consider the second-order, degree-1 recurrence

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

Problem

Given $u_{-1}, u_0 \in \mathbb{Q}$ (or $\overline{\mathbb{Q}} \cap \mathbb{R}$), we want to decide whether the corresponding full sequence $\langle u_n \rangle_{n=-1}^{\infty}$ is *minimal* and/or *positive*.

- For each n , consider the characteristic polynomial $(2n + 3)z^2 - (6n + 7)z + (4n + 1)$ and denote the roots λ_n and Λ_n
- Observe that $\lambda_n \rightarrow \lambda = 1$ and $\Lambda_n \rightarrow \Lambda = 2$ as $n \rightarrow \infty$
- By the Poincaré-Perron theorem, for any solution sequence $\langle u_n \rangle_n$, we have either $u_n/u_{n-1} \rightarrow \lambda$ or $u_n/u_{n-1} \rightarrow \Lambda$
 - Clearly $\langle u_n \rangle_n$ is minimal iff $u_n/u_{n-1} \rightarrow \lambda$

Approximating the Critical Ratio μ

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

Approximating the Critical Ratio μ

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

$$x_n = \left(\frac{6n + 7}{2n + 3} \right) x_{n-1} + \left(-\frac{4n + 1}{2n + 3} \right) x_{n-2}$$

Approximating the Critical Ratio μ

$$(2n + 3)x_n = (6n + 7)x_{n-1} - (4n + 1)x_{n-2}$$

$$x_n = \left(\frac{6n + 7}{2n + 3}\right) x_{n-1} + \left(-\frac{4n + 1}{2n + 3}\right) x_{n-2}$$

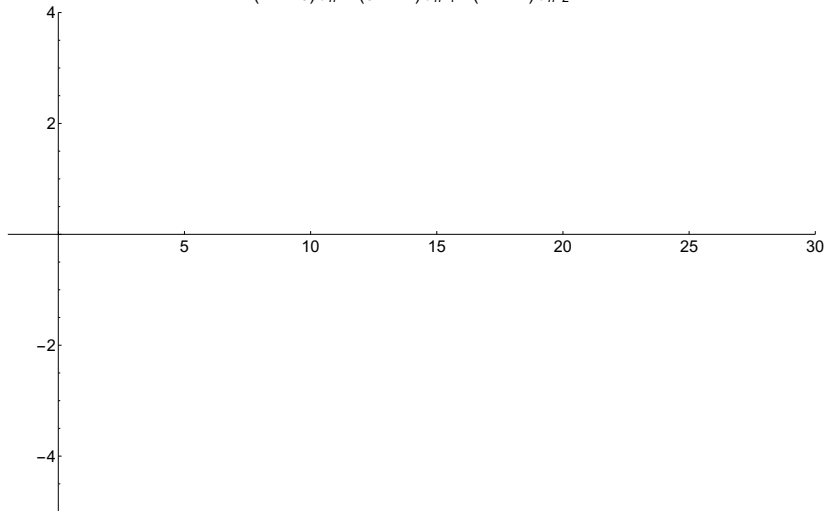
Pincherle's theorem:

$$\mu = -\prod_{n=1}^{\infty} \frac{\left(-\frac{4n+1}{2n+3}\right)}{\left(\frac{6n+7}{2n+3}\right)} = -\frac{1}{13/5 + \frac{-9/7}{19/7 + \frac{-13/9}{25/9 + \frac{-17/11}{31/11 + \dots}}}}$$

$\approx 0.511238 \dots$

Plotting (u_n/u_{n-1}) against n

$$(2n+3)u_n = (6n+7)u_{n-1} - (4n+1)u_{n-2}$$



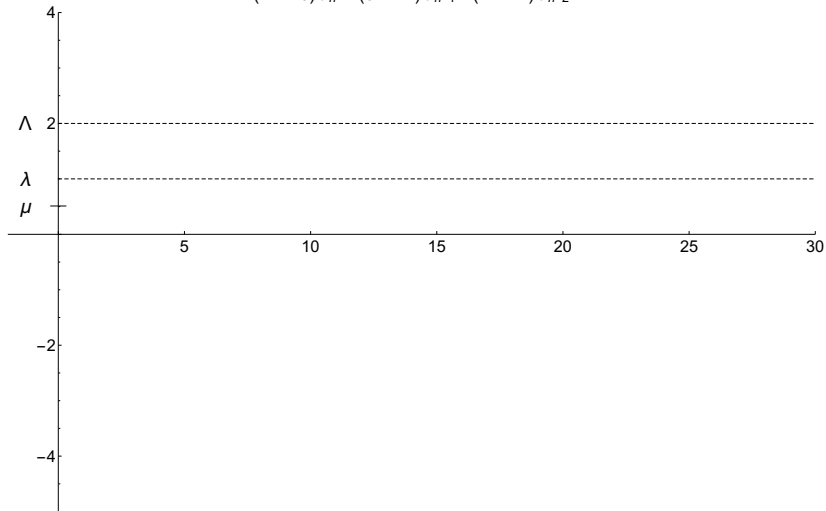
$$\mu \approx 0.511238 \dots$$

$$\lambda = 1$$

$$\Lambda = 2$$

Plotting (u_n/u_{n-1}) against n

$$(2n+3)u_n = (6n+7)u_{n-1} - (4n+1)u_{n-2}$$



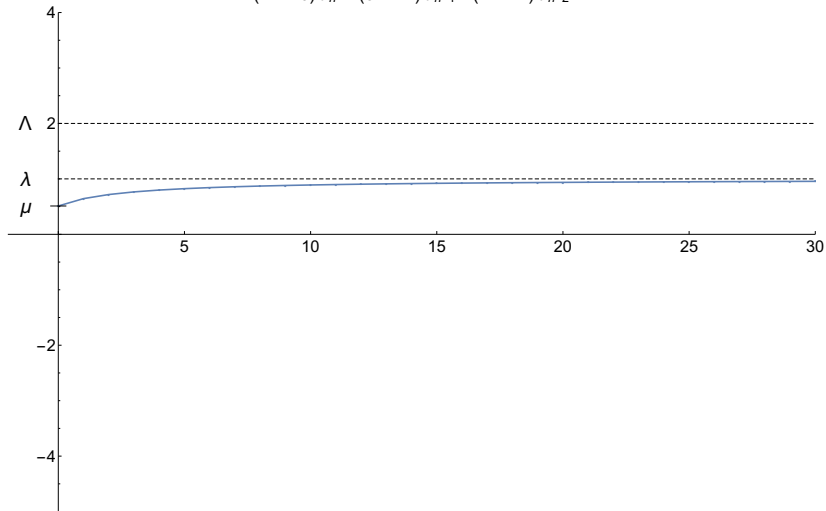
$$\mu \approx 0.511238 \dots$$

$$\lambda = 1$$

$$\Lambda = 2$$

Plotting (u_n/u_{n-1}) against n

$$(2n+3)u_n = (6n+7)u_{n-1} - (4n+1)u_{n-2}$$



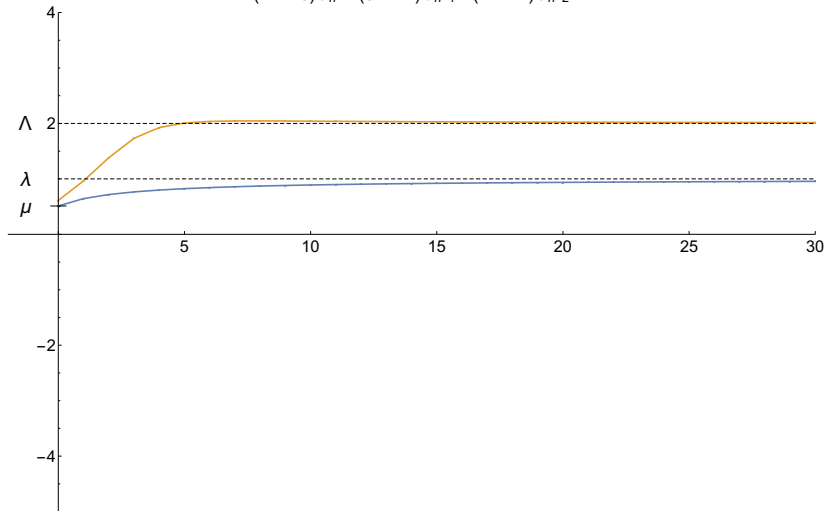
$$\mu \approx 0.511238 \dots$$

$$\lambda = 1$$

$$\Lambda = 2$$

Plotting (u_n/u_{n-1}) against n

$$(2n+3)u_n = (6n+7)u_{n-1} - (4n+1)u_{n-2}$$



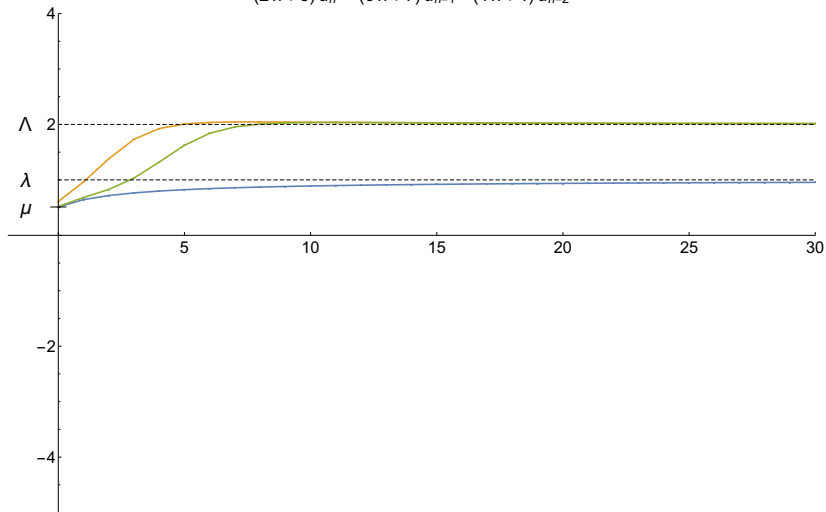
$$\mu \approx 0.511238 \dots$$

$$\lambda = 1$$

$$\Lambda = 2$$

Plotting (u_n/u_{n-1}) against n

$$(2n+3)u_n = (6n+7)u_{n-1} - (4n+1)u_{n-2}$$



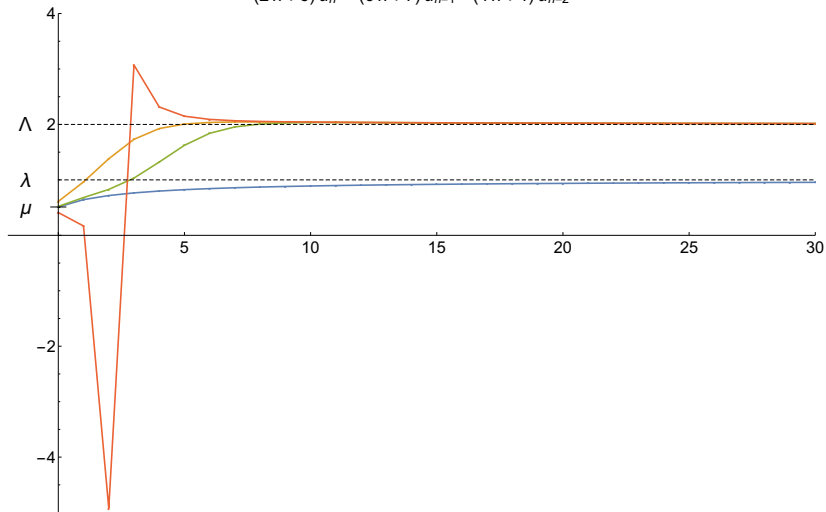
$$\mu \approx 0.511238\dots$$

$$\lambda = 1$$

$$\Lambda = 2$$

Plotting (u_n/u_{n-1}) against n

$$(2n+3)u_n = (6n+7)u_{n-1} - (4n+1)u_{n-2}$$

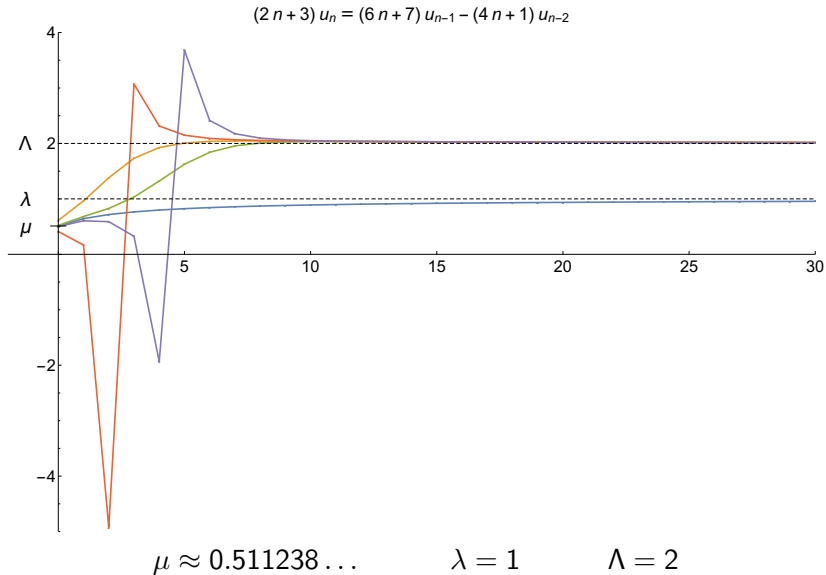


$$\mu \approx 0.511238 \dots$$

$$\lambda = 1$$

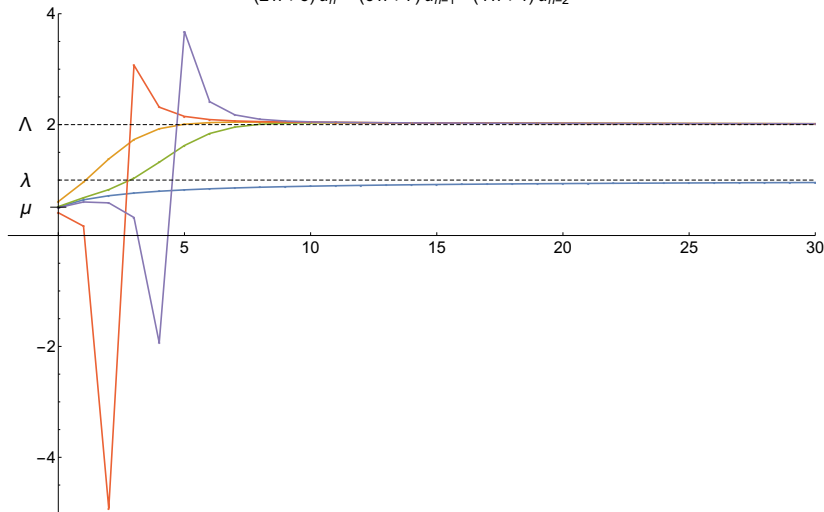
$$\Lambda = 2$$

Plotting (u_n/u_{n-1}) against n



Plotting (u_n/u_{n-1}) against n

$$(2n+3)u_n = (6n+7)u_{n-1} - (4n+1)u_{n-2}$$



$$\text{Let } f(x) = \sum_{n=0}^{\infty} u_{n-1} x^n$$

Singularity Analysis and Minimality

Let $f(x) = \sum_{n=0}^{\infty} u_{n-1} x^n$

Singularity Analysis and Minimality

$$\text{Let } f(x) = \sum_{n=0}^{\infty} u_{n-1} x^n$$

$$f'(x) + \frac{9x^2 - 13x + 3}{4x^3 - 6x^2 + 2x} f(x) = \frac{-13u_{-1}x + 5u_0x + 3u_{-1}}{4x^3 - 6x^2 + 2x}$$

Singularity Analysis and Minimality

$$\text{Let } f(x) = \sum_{n=0}^{\infty} u_{n-1} x^n$$

$$f'(x) + \frac{9x^2 - 13x + 3}{4x^3 - 6x^2 + 2x} f(x) = \frac{-13u_{-1}x + 5u_0x + 3u_{-1}}{4x^3 - 6x^2 + 2x}$$

$$\Rightarrow f(x) = \frac{\sqrt{1-x} \int_0^x \frac{\left(\frac{1}{2}-y\right)^{1/4} \sqrt{y}(3u_{-1}+(-13u_{-1}+5u_0)y)}{4(1-y)^{3/2}} dy}{x^{3/2} \left(\frac{1}{2}-x\right)^{5/4}}$$

Singularity Analysis and Minimality

$$\text{Let } f(x) = \sum_{n=0}^{\infty} u_{n-1} x^n$$

$$f'(x) + \frac{9x^2 - 13x + 3}{4x^3 - 6x^2 + 2x} f(x) = \frac{-13u_{-1}x + 5u_0x + 3u_{-1}}{4x^3 - 6x^2 + 2x}$$

$$\Rightarrow f(x) = \frac{\sqrt{1-x} \int_0^x \frac{(\frac{1}{2}-y)^{1/4} \sqrt{y}(3u_{-1}+(-13u_{-1}+5u_0)y)}{4(1-y)^{3/2}} dy}{x^{3/2} \left(\frac{1}{2} - x\right)^{5/4}}$$

Singularity Analysis and Minimality

$$\text{Let } f(x) = \sum_{n=0}^{\infty} u_{n-1} x^n$$

$$f'(x) + \frac{9x^2 - 13x + 3}{4x^3 - 6x^2 + 2x} f(x) = \frac{-13u_{-1}x + 5u_0x + 3u_{-1}}{4x^3 - 6x^2 + 2x}$$

$$\Rightarrow f(x) = \frac{\sqrt{1-x} \int_0^x \frac{\left(\frac{1}{2}-y\right)^{1/4} \sqrt{y}(3u_{-1}+(-13u_{-1}+5u_0)y)}{4(1-y)^{3/2}} dy}{x^{3/2} \left(\frac{1}{2}-x\right)^{5/4}}$$

Singularity Analysis and Minimality

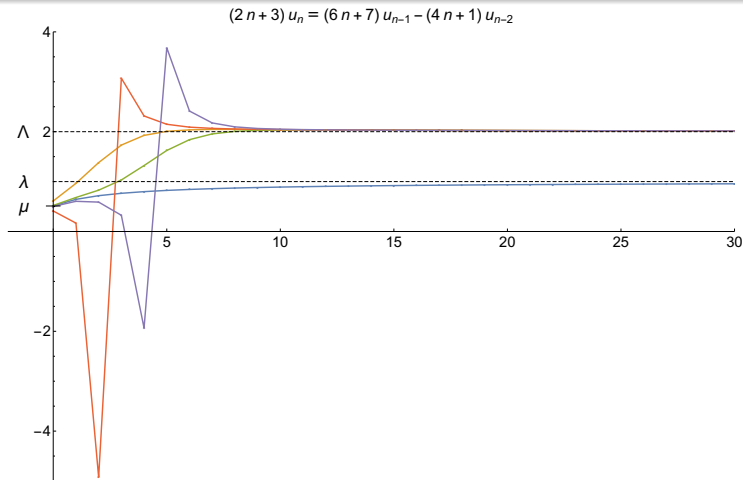
$$\text{Let } f(x) = \sum_{n=0}^{\infty} u_{n-1} x^n$$

$$f'(x) + \frac{9x^2 - 13x + 3}{4x^3 - 6x^2 + 2x} f(x) = \frac{-13u_{-1}x + 5u_0x + 3u_{-1}}{4x^3 - 6x^2 + 2x}$$

$$\Rightarrow f(x) = \frac{\sqrt{1-x} \int_0^x \frac{(\frac{1}{2}-y)^{1/4} \sqrt{y}(3u_{-1}+(-13u_{-1}+5u_0)y)}{4(1-y)^{3/2}} dy}{x^{3/2} \left(\frac{1}{2} - x\right)^{5/4}}$$

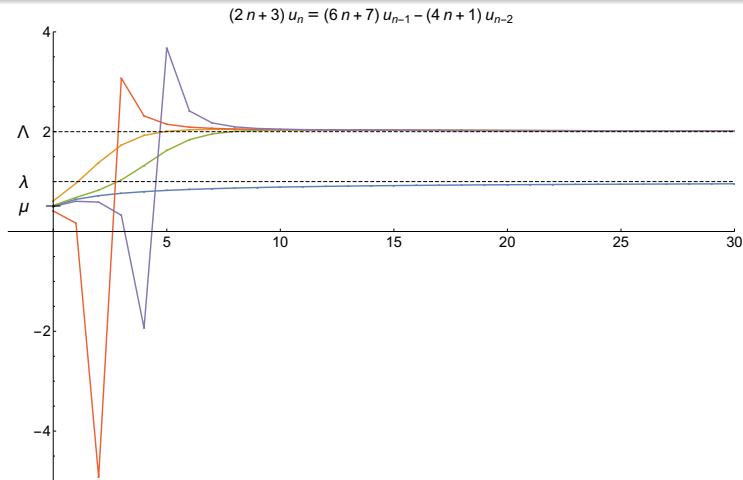
$$\langle u_n \rangle_n \text{ is minimal iff } \int_0^{1/2} \frac{(\frac{1}{2}-y)^{1/4} \sqrt{y}(3u_{-1}+(-13u_{-1}+5u_0)y)}{4(1-y)^{3/2}} dy = 0 \quad \square$$

Plotting (u_n/u_{n-1}) against n



$\langle u_n \rangle_n$ is minimal iff $\int_0^{1/2} \dots dy = 0$

Plotting (u_n/u_{n-1}) against n



$\langle u_n \rangle_n$ is minimal	iff	$\int_0^{1/2} \dots dy = 0$
$\langle u_n \rangle_n$ is positive	iff	$\langle u_n \rangle_n$ is minimal or $u_0/u_{-1} > \mu$

Asymptotics and Inequalities: Summary

Consider holonomic sequences over real algebraic numbers:

Asymptotics and Inequalities: Summary

Consider holonomic sequences over real algebraic numbers:

Theorem

For second-order sequences, the Positivity Problem reduces to (one or more instances of) the Minimality Problem.

Asymptotics and Inequalities: Summary

Consider holonomic sequences over real algebraic numbers:

Theorem

For second-order sequences, the Positivity Problem reduces to (one or more instances of) the Minimality Problem.

Theorem

For second-order, degree-1 sequences having rational roots (simple or repeated), the Positivity and Minimality Problems reduce to whether a 1-period vanishes, and is therefore decidable.

Asymptotics and Inequalities: Summary

Consider holonomic sequences over real algebraic numbers:

Theorem

For second-order sequences, the Positivity Problem reduces to (one or more instances of) the Minimality Problem.

Theorem

For second-order, degree-1 sequences having rational roots (simple or repeated), the Positivity and Minimality Problems reduce to whether a 1-period vanishes, and is therefore decidable.

- Irrational-roots instances of the above, as well as the general Hypergeometric Inequality Problem, can be reduced to equality of special functions (hypergeometric and beta functions) evaluated at real-algebraic points. These do not (at first blush) appear to be periods or exponential periods.

Asymptotics and Inequalities: Summary

Consider holonomic sequences over real algebraic numbers:

Theorem

For second-order sequences, the Positivity Problem reduces to (one or more instances of) the Minimality Problem.

Theorem

For second-order, degree-1 sequences having rational roots (simple or repeated), the Positivity and Minimality Problems reduce to whether a 1-period vanishes, and is therefore decidable.

- Irrational-roots instances of the above, as well as the general Hypergeometric Inequality Problem, can be reduced to equality of special functions (hypergeometric and beta functions) evaluated at real-algebraic points. These do not (at first blush) appear to be periods or exponential periods.
- Positivity and Minimality for second-order holonomic sequences of higher degrees are not even known to reduce to a vanishing-integral condition. This remains wide open.

- Vastly underexplored territory:

Algorithmic treatment of analytic properties of holonomic systems!

Outlook

- Vastly underexplored territory:

Algorithmic treatment of analytic properties of holonomic systems!

