Fragments of Hilbert's Program

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(joint work with Valérie Berthé, Florian Luca, James Worrell,
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erc European Research Council
DynAMiCs
Funded by DFG Deutsche Forschungsgemeinschaft German Research Foundation

Calculemus!

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If I have seen farther than others, it is because I have stood on the shoulders of giants;

Calculemus!

If I have seen farther than others, it is because I have stood on the shoulders of giants; You, my dear Hooke, have not.

Leibniz's Dream: reducing mathematics to mere computation

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''In large language models (LLMs), a token is a chunk of text used as
a basic unit for processing -- typically a word, subword, or even
character, depending on the language and tokenizer.  The model reads
and generates text in terms of these tokens, not full words or
sentences, enabling it to handle diverse languages and structures
efficiently.''
```

Hilbert's Program and the Entscheidungsproblem

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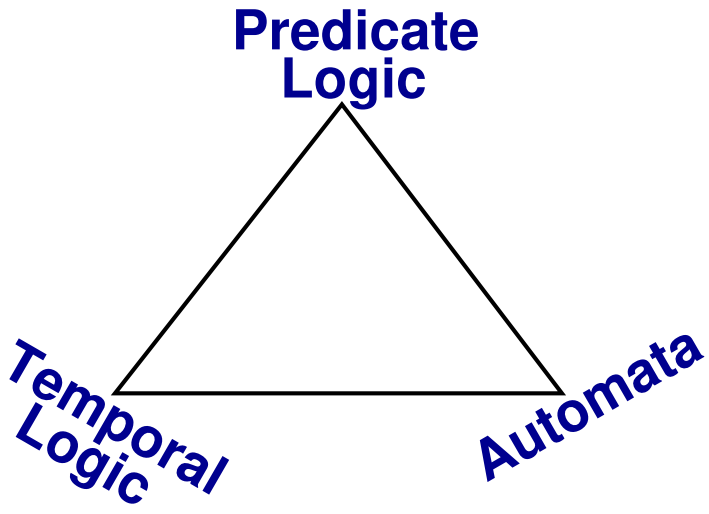
Wir müssen wissen. Wir werden wissen.

Fast forward about a hundred years. . .

The Holy Trinity of Theoretical Computer Science



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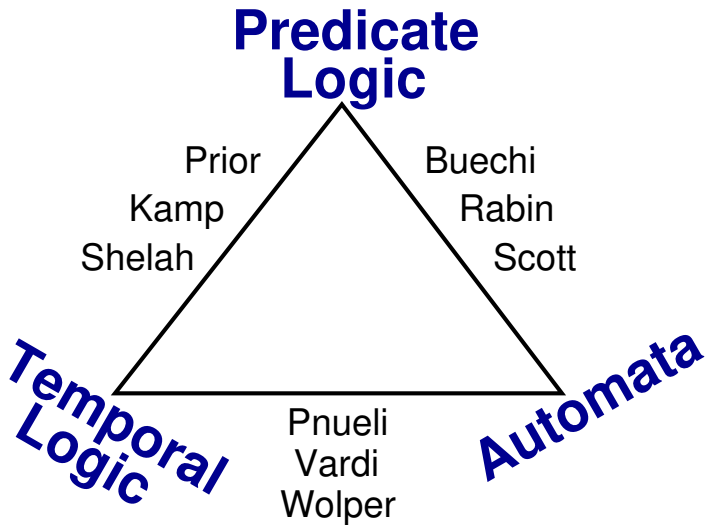

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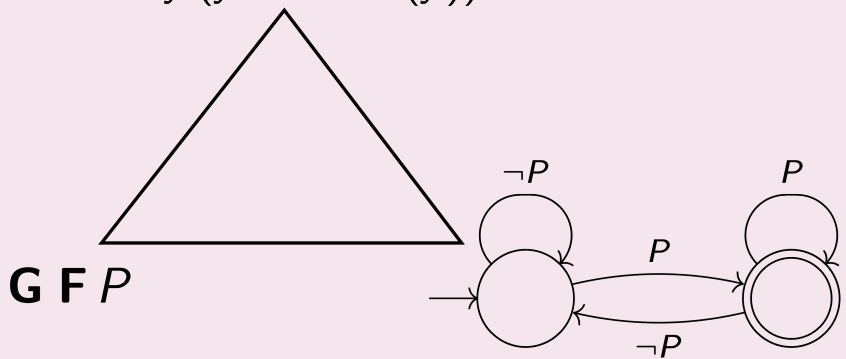
$$\forall x \, \exists y \, (y > x \wedge P(y))$$



The Monadic Second-Order Logic of Order

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$$\varphi ::= x < y \mid P(x) \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi \mid \exists P\, \varphi$$

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Examples:

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Theorem (Dirichlet, 1837)

Let PRIMES denote the set of prime numbers. Then

$$\forall x \exists y\, (y > x \wedge \mathit{PRIMES}(y) \wedge \exists Q\, .\, ((*) \wedge Q(y)))\, .$$

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- Above formula is an example of a sentence in the MSO theory of $\langle \mathbb{N}; <, \mathit{PRIMES} \rangle$

Question (Büchi and Landweber, 1969)

Is the MSO theory of $\langle \mathbb{N}; <, PRIMES \rangle$ *decidable?*

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Theorem (Bateman, Jockusch, and Woods, 1993)

Yes, assuming Schinzel's Hypothesis H.

A glimpse of the Hilbert Landscape

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Solution requires Baker's theorem on linear forms in logarithms!

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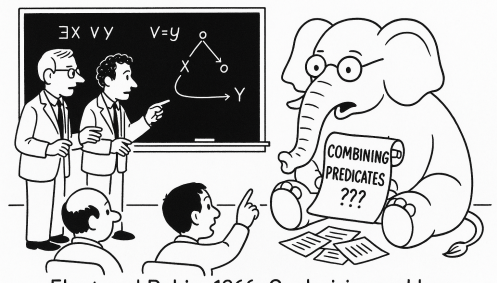

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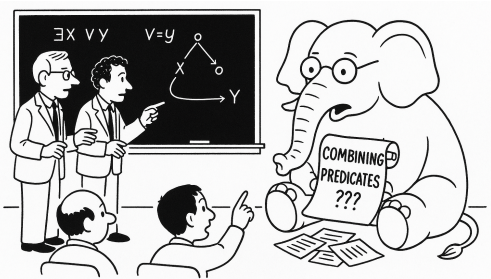

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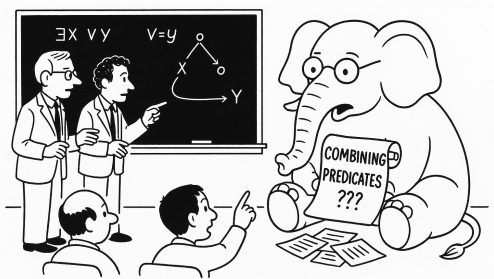

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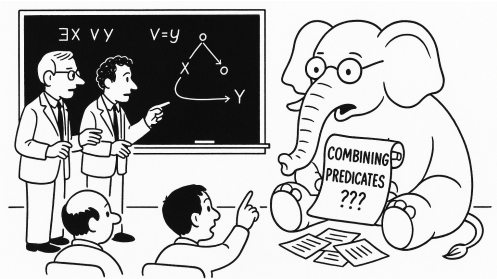

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...much subsequent work over the ensuing decades (Landweber, Semenov, Thomas, Rabinovich, Carton, ...)

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RESEARCH-ARTICLE | OPEN ACCESS

On the Decidability of Monadic Second-Order Logic with Arithmetic Predicates

Authors: Valérie Berthé, Toghrul Karimov, Joris Nieuwveld, Joël Ouaknine, Mihir Vahanwala, and James Worrell | Authors Info & Claims

LICS '24: Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science

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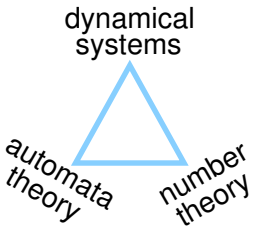

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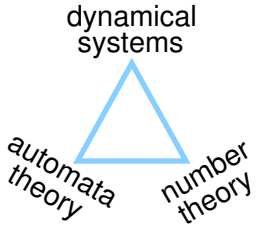

We are planning to implement this algorithm!

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There are 26 pairs in total;
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Yes, there are infinitely many.
The first pair is
$(m = 1788, n = 1128)$;
3^{1128} has 539 digits!

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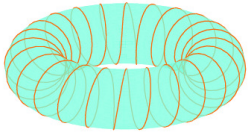
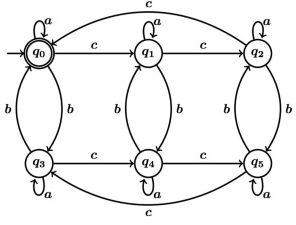
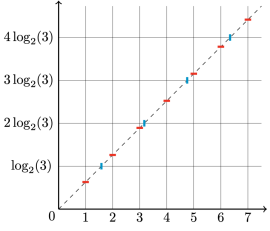


logic & automata theory



Diophantine geometry



Normality and Disjunctivity

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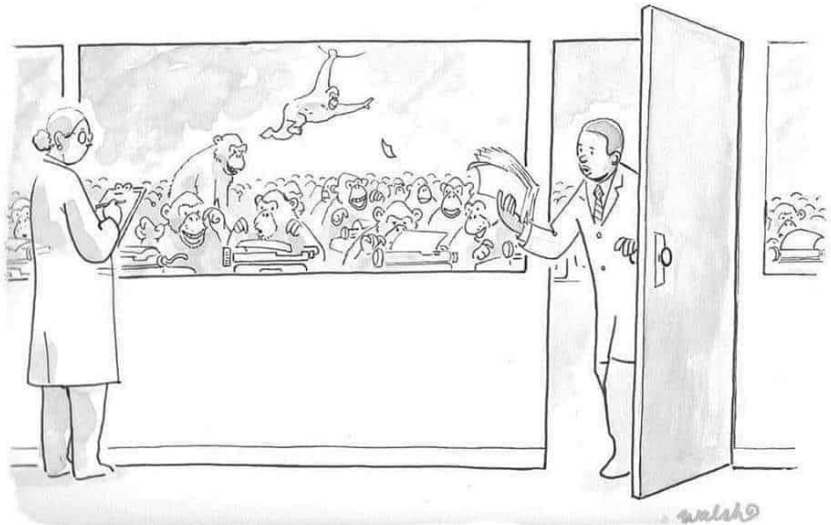
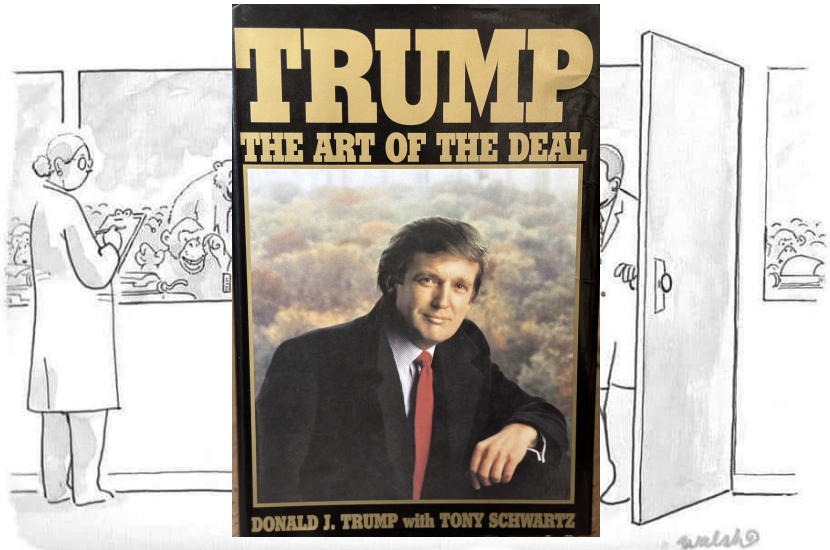


"No Shakespeare yet, but here's another copy of 'Art of the Deal'."



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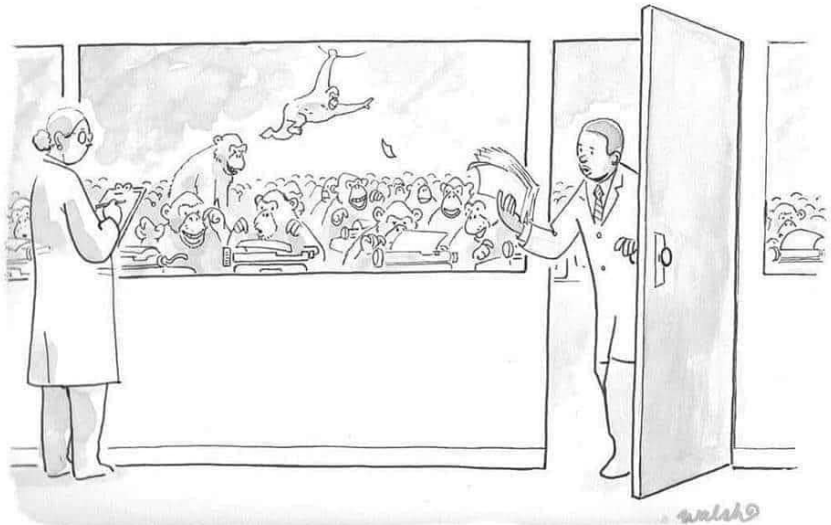

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There are 26 pairs in total;
the last one is $(m = 8, n = 5)$,
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Yes, there are infinitely many.
The first pair is
$(m = 1788, n = 1128)$;
3^{1128} has 539 digits!

Are there finitely many n such that the number of perfect squares between 2^n and 2^{n+1} is even, *and* the number of perfect squares between 2^{n+1} and 2^{n+2} is divisible by 3?

Theorem (Berthé *et al.*, LICS 2024)

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This is open! However:

- If $\sqrt{2}$ is disjunctive in binary, then there are infinitely many such n;
- If certain specific strings only occur *finitely often* in the binary expansion of $\sqrt{2}$, then there are only finitely many such n

Fragments of Hilbert's Program: More open problems

Open Problems

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The *Brocard-Ramanujan Problem:* Find all integers m and n such that

$$n! + 1 = m^2$$

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- At the very least, would imply *effective* lower bounds on sums of S-units

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Theorem (folklore)

The FO theory of $\langle \mathbb{N}; 0, 1, <, +, SQUARES \rangle$ *is undecidable.*

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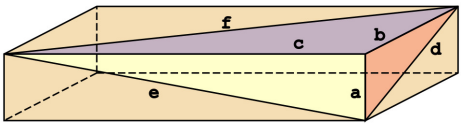

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Suppose that $\langle s_1, s_2, \ldots, s_M \rangle$ is an increasing sequence of *consecutive* perfect squares. Then for all $1 \leq n \leq M - 2$,

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Some open questions

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Büchi's Problem for perfect cubes (and arbitrary non-linear polynomials)

Suppose that $\langle t_1, t_2, \ldots, t_M \rangle$ *is an increasing sequence of perfect cubes such that, for all* $1 \leq n \leq M-3$,

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Is there some value of M *which entails that the cubes are necessarily consecutive?*
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Is the two-variable first-order fragment of $\langle \mathbb{Z}; 0, 1, <, +, -, SQUARES \rangle$ *decidable?*

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i.e., is there an algorithm to determine whether a given formula $\exists\, x, y\,.\,\varphi(x, y)$ *is true?*

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More open questions and ongoing work

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Theorem (Bacik, Nieuwveld, O., Vahanwala, Wieser, Worrell, 2025 — in preparation)

Let $P_1, \ldots, P_k$ be univariate polynomial predicates of degree at most 3.
Then $\mathrm{FO}^1\langle \mathbb{Z}; 0, 1, <, +, -, P_1, \ldots, P_k \rangle$ is decidable.

Two further open questions...

Weak expansions of linear arithmetic over the rationals

What about all of the above questions over $\mathbb{Q}$ rather than $\mathbb{Z}$?

Two further open questions...

Weak expansions of linear arithmetic over the rationals

What about all of the above questions over $\mathbb{Q}$ rather than $\mathbb{Z}$?

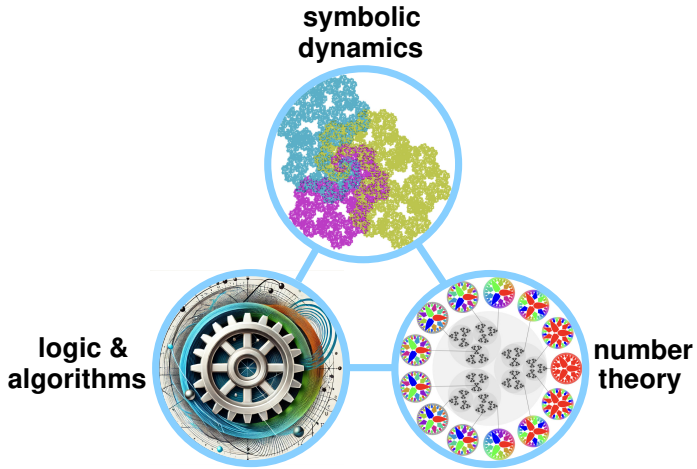
Efficient implementation

What is the computational complexity of all these algorithms?

Outlook

Hilbert's legacy continues to be a rich source of inspiration!

Thriving research endeavour at the confluence of computer science and mathematics



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